



Bandyopadhyay, **Beattie**+2025 (ApJL). *Density Fluctuation–Mach Number Scaling in Compressible, High Plasma Beta Turbulence: In Situ Space Observations and High-Reynolds Number Simulations*

Fundamental Results from the World's Largest Simulation of Compressible MHD Turbulence

AAPPS-DPP 2025

James Beattie

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Princeton / Canadian Institute for Theoretical Astrophysics

In collaboration: Amitava Bhattacharjee (Princeton), Christoph Federrath (ANU), Riddhi Bandyopadhyay (Princeton)

Beattie+2025 (*Nature Astronomy*). The spectrum of magnetized turbulence in the interstellar medium

Beattie & Bhattacharjee (submitted *PRL*). Scale-dependent alignment in compressible magnetohydrodynamic turbulence

Fact: the galaxy that we reside in is in a state of highly-compressible fluid turbulence

Supernovae alone provide enough energy flux to fuel a continuously driven cascade

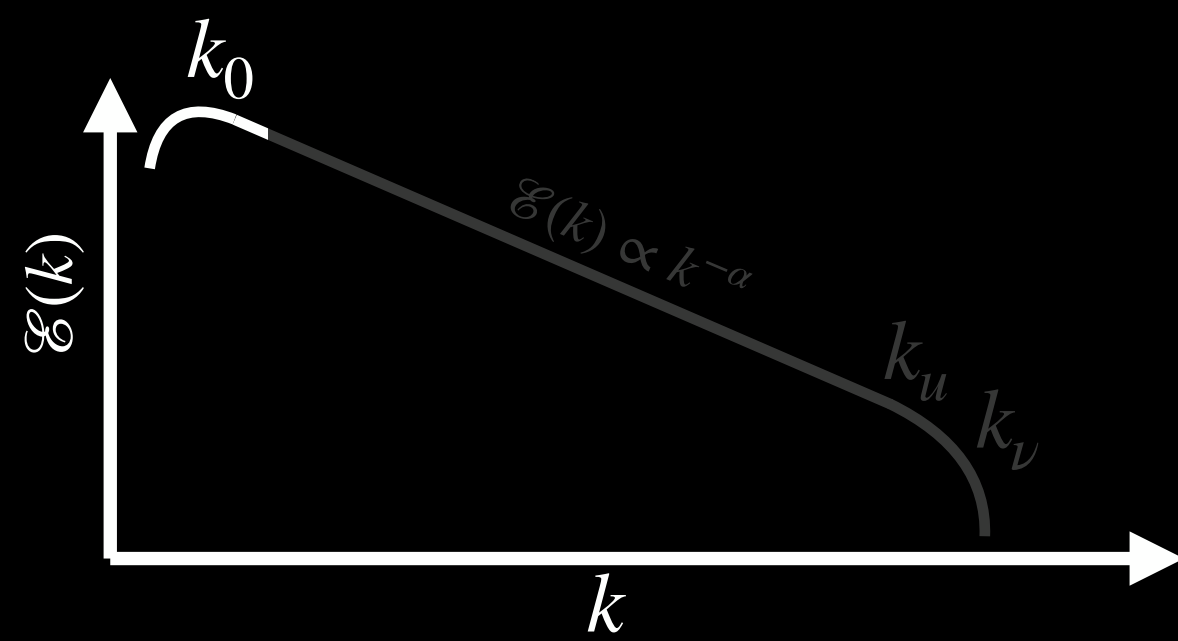
Beattie + (2025; ApJ) So long Kolmogorov: the forward and backwards cascades in a supernovae-driven, multiphase ISM

Connor, **Beattie** + (2025, submitted ApJ) Cascading from the winds to the disk: universality of supernovae-driven turbulence in different galactic ISMs

Beattie (2025; submitted ApJL) Supernovae drive large-scale incompressible turbulence from small-scale instabilities

Simulation from: Goldbaum, N. J., Krumholz, M. R., & Forbes, J. C. 2016

Turbulence Cascade



Credit: NASA, ESA, S. Beckwith (STScI) and the Hubble Heritage Team (STScI/AURA)

inspired by Shukurov, 2011

$$\begin{aligned} \text{WIM: } \text{Re} &\sim 10^7 & \lambda_{\text{mfp}} &\sim 5 R_{\oplus} \\ \text{WNM: } \text{Re} &\sim 10^7 & \lambda_{\text{mfp}} &\sim 1.3 \text{ AU} \\ \text{CNM: } \text{Re} &\sim 10^{10} & \lambda_{\text{mfp}} &\sim 0.3 \text{ pc} \end{aligned}$$

Ferrière, 2020; Plasma Physics and Controlled Fusion

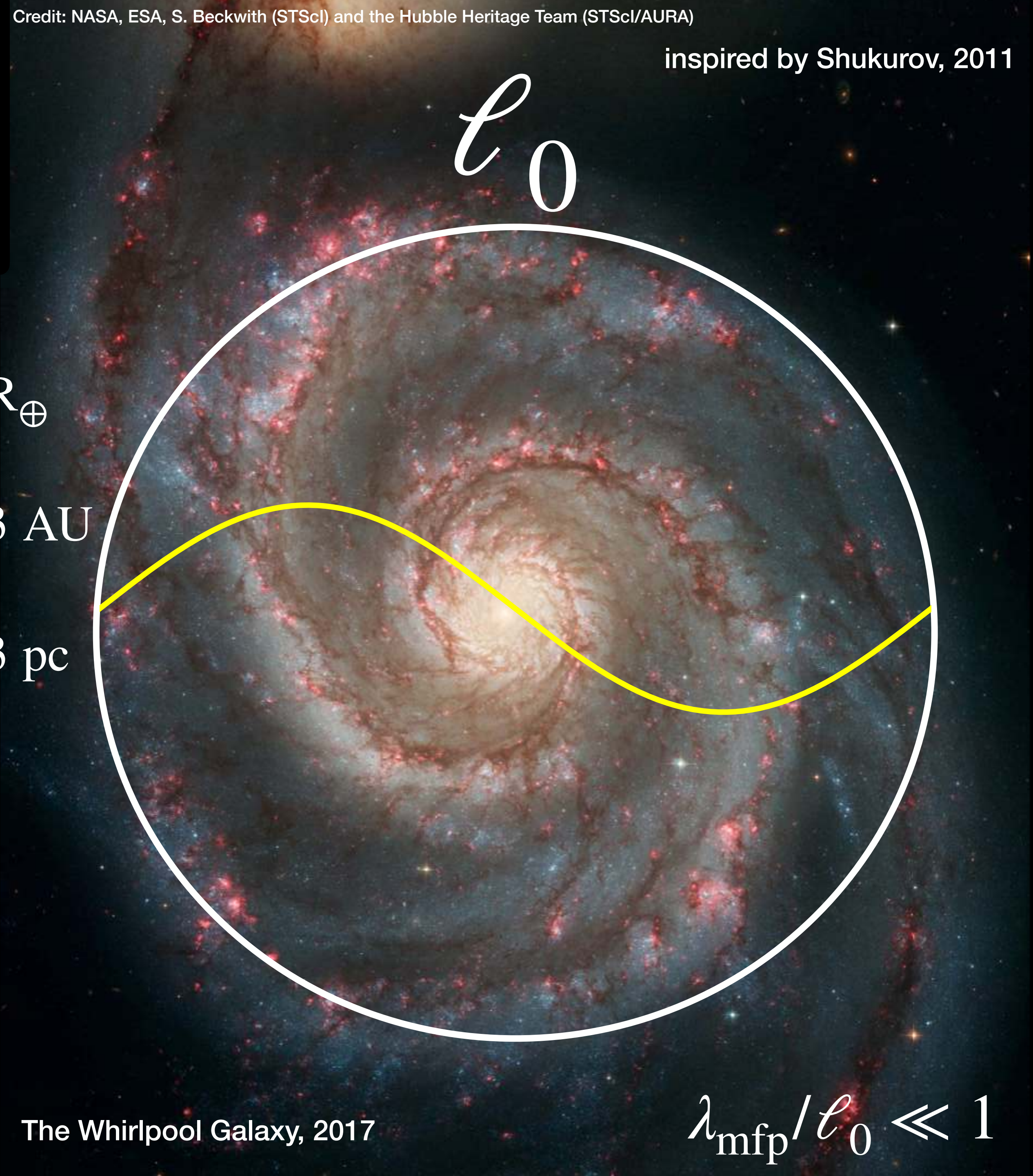
The quadratic nonlinear term

$$|\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u})|$$

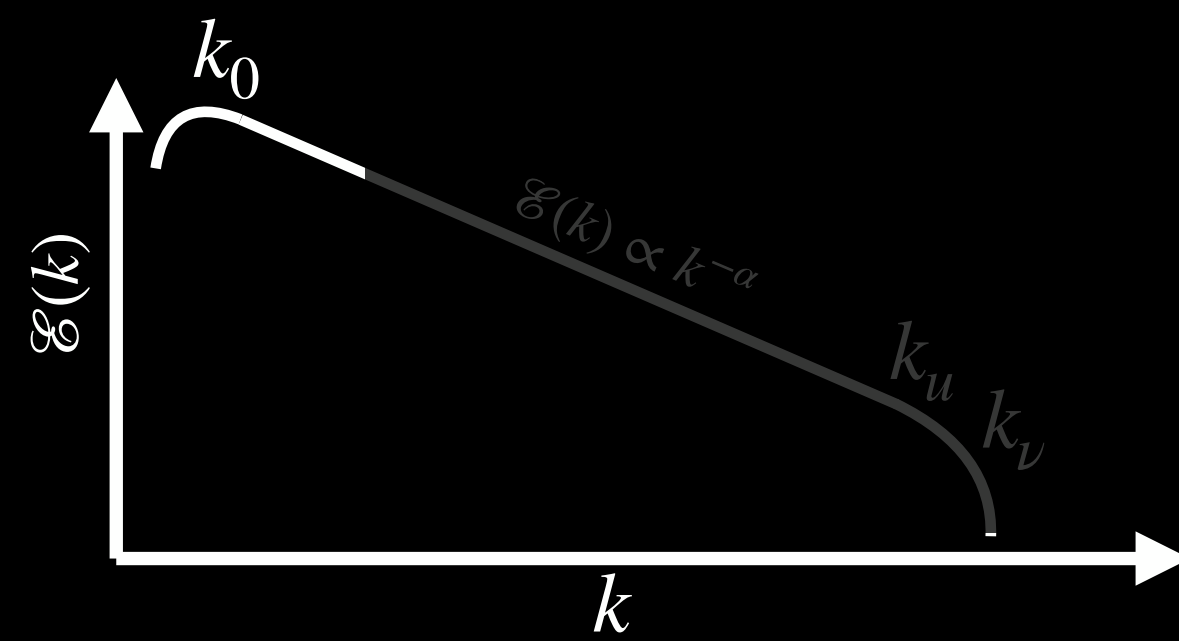
dominates on large scales, ℓ

The Whirlpool Galaxy, 2017

$$\lambda_{\text{mfp}} / \ell_0 \ll 1$$



Turbulence Cascade



The quadratic nonlinear term

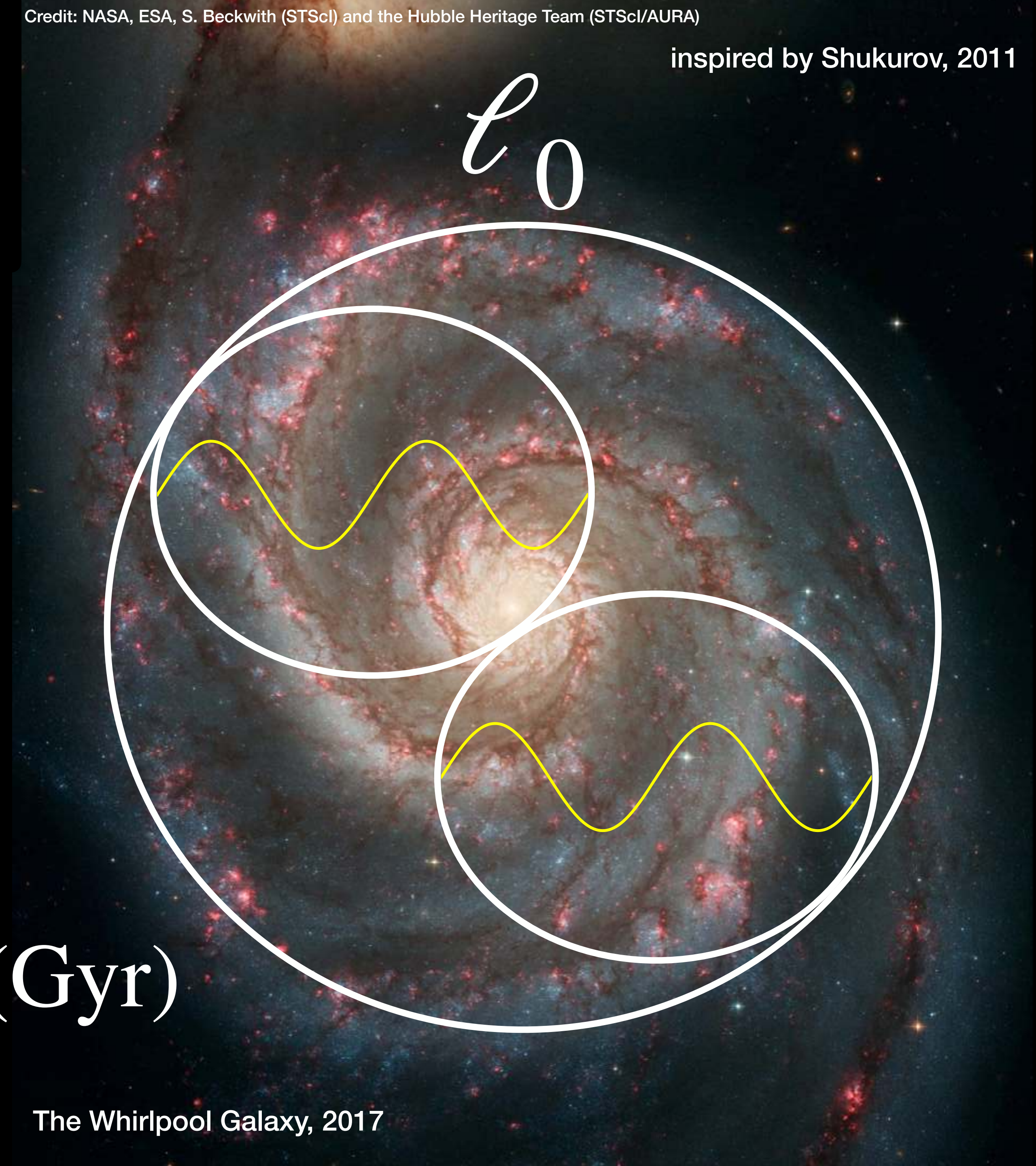
$$|\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u})|$$

dominates on large scales, ℓ

Creates new modes on

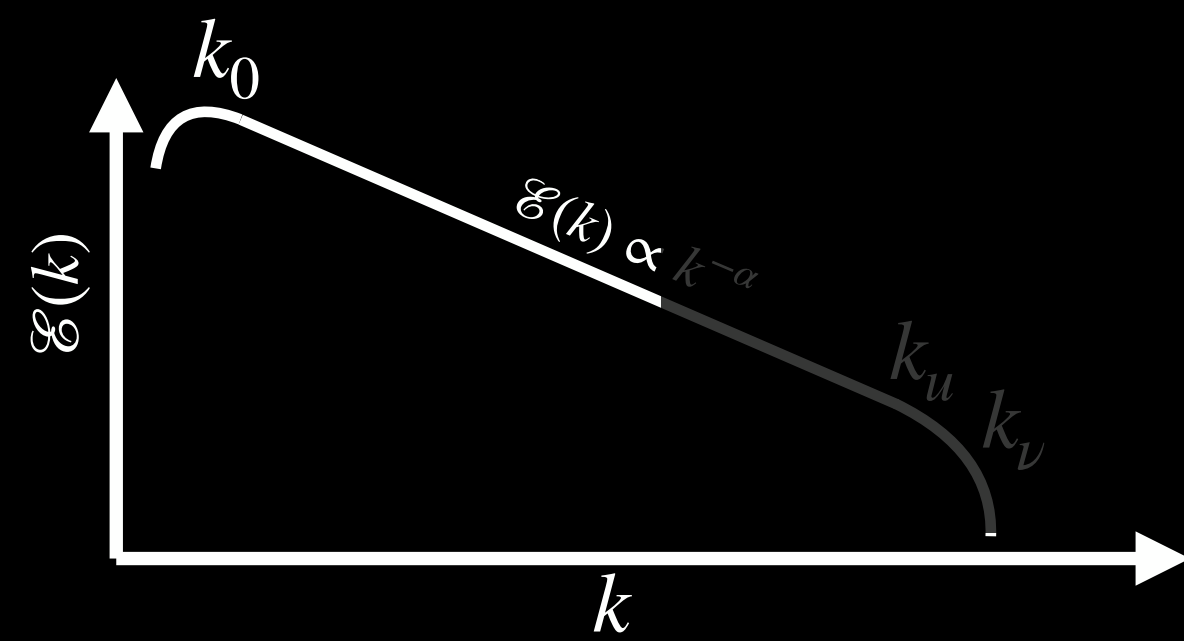
$$t_{\text{nl}} \sim [\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u})]^{-1} \sim \mathcal{O}(\text{Gyr})$$

$$\dot{\varepsilon} \sim u_0^3 / \ell_0$$



The Whirlpool Galaxy, 2017

Turbulence Cascade



inside of the cascade

Credit: NASA, ESA, S. Beckwith (STScI) and the Hubble Heritage Team (STScI/AURA)

inspired by Shukurov, 2011

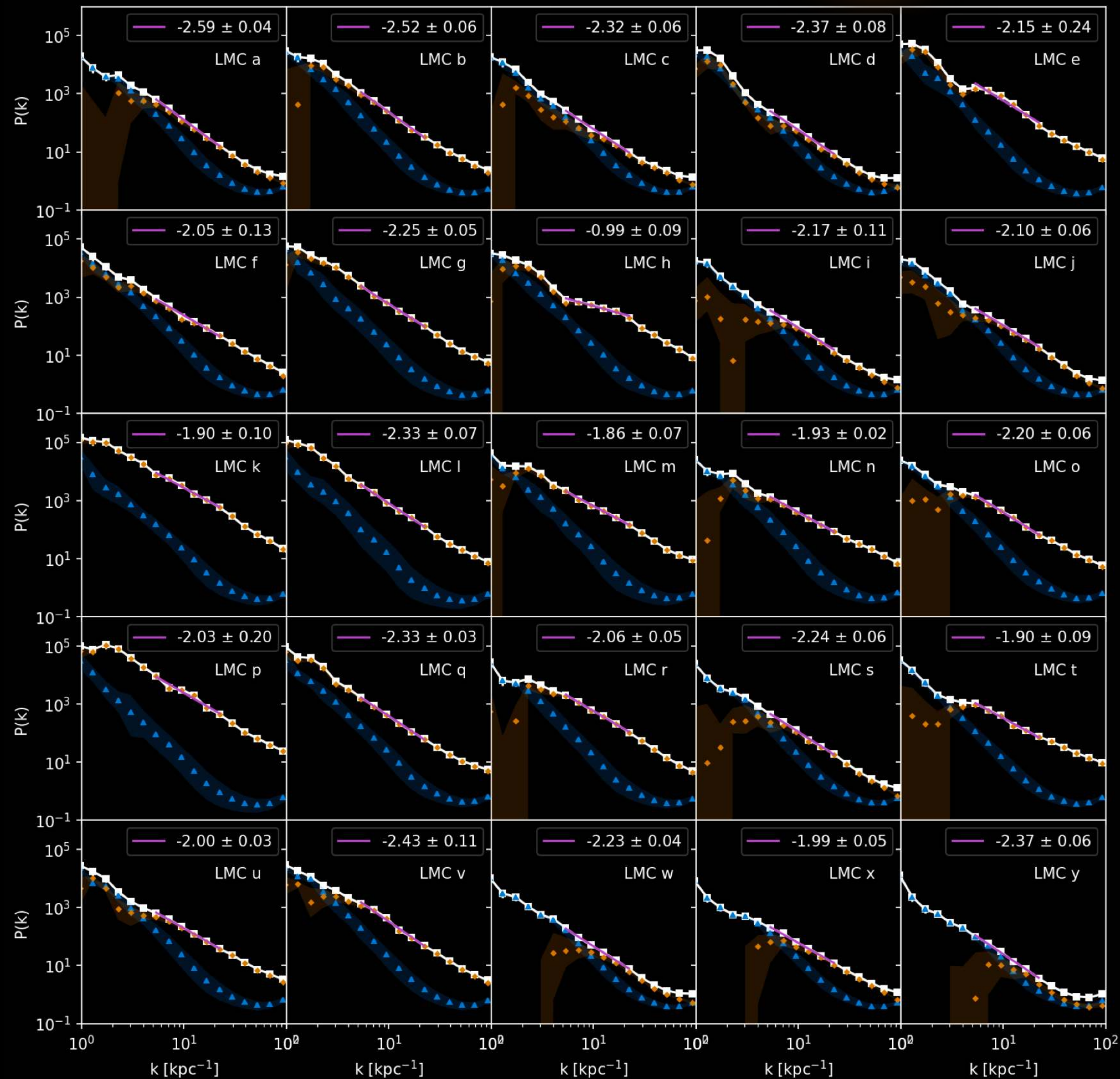


The Whirlpool Galaxy, 2017

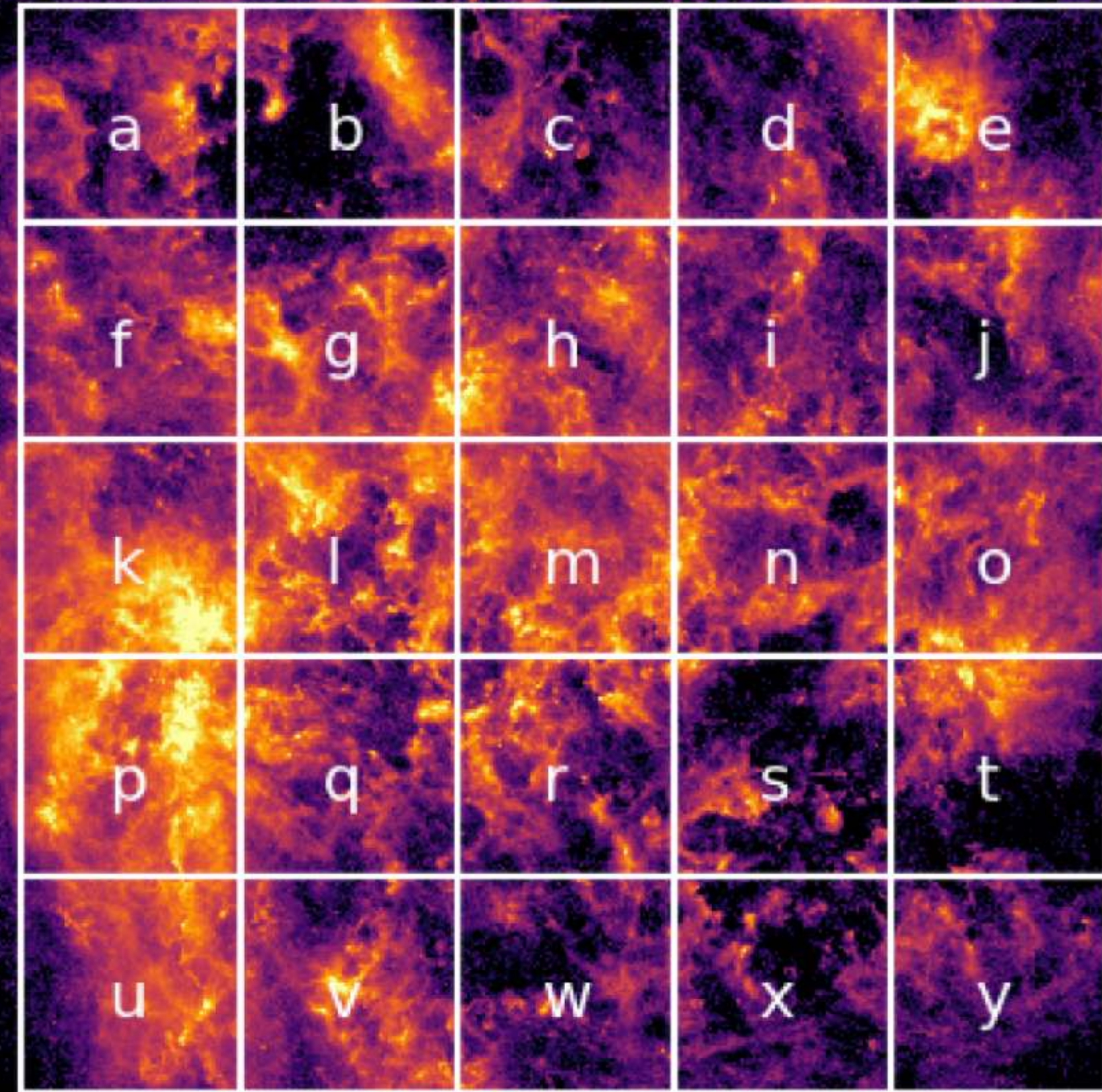
Turbulence



LMC: $500\mu\text{m}$, *Herschel* (processing Gordon et al. 2014)



1 kpc

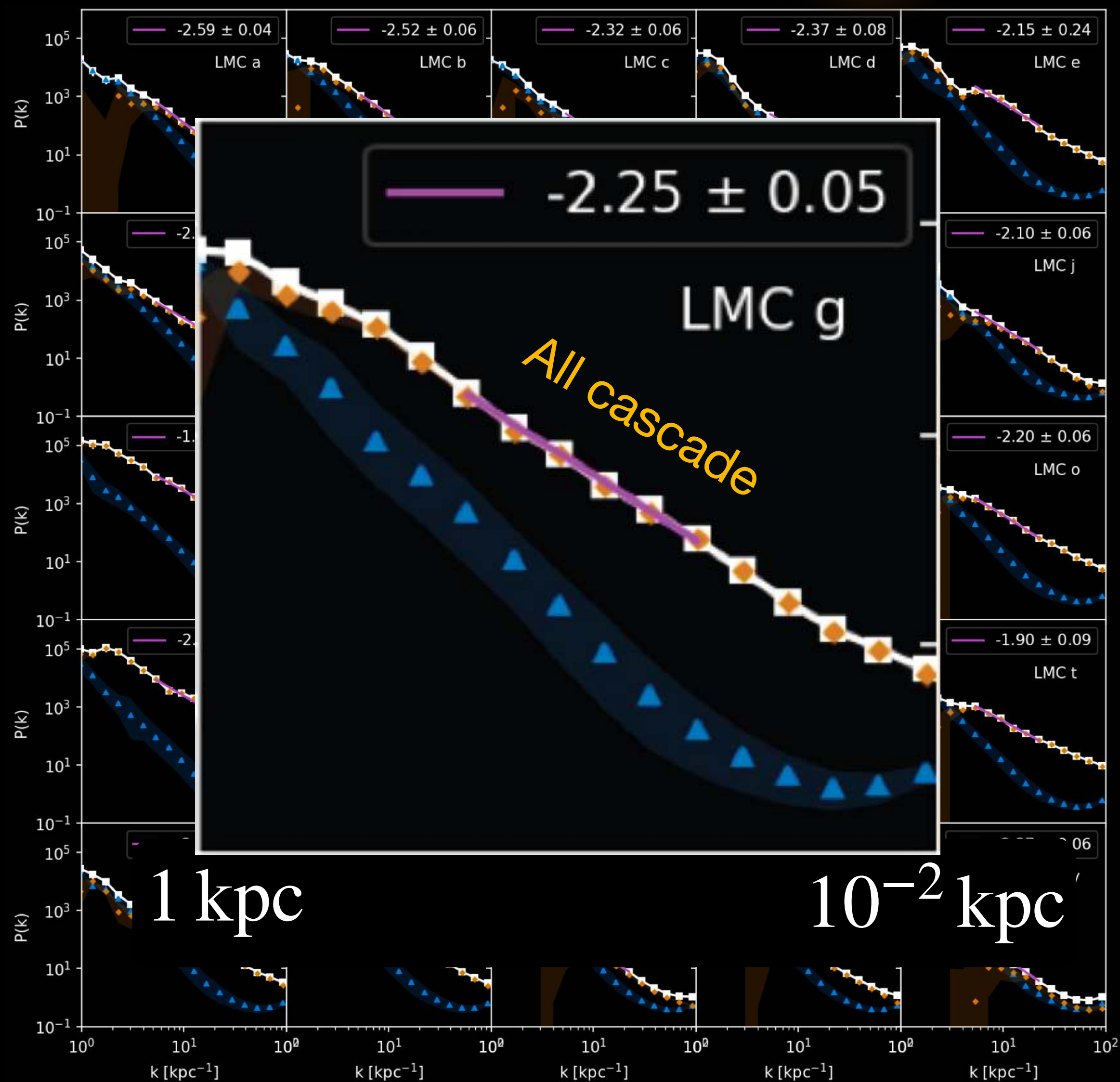


Colman, et al., 2022, MNRAS

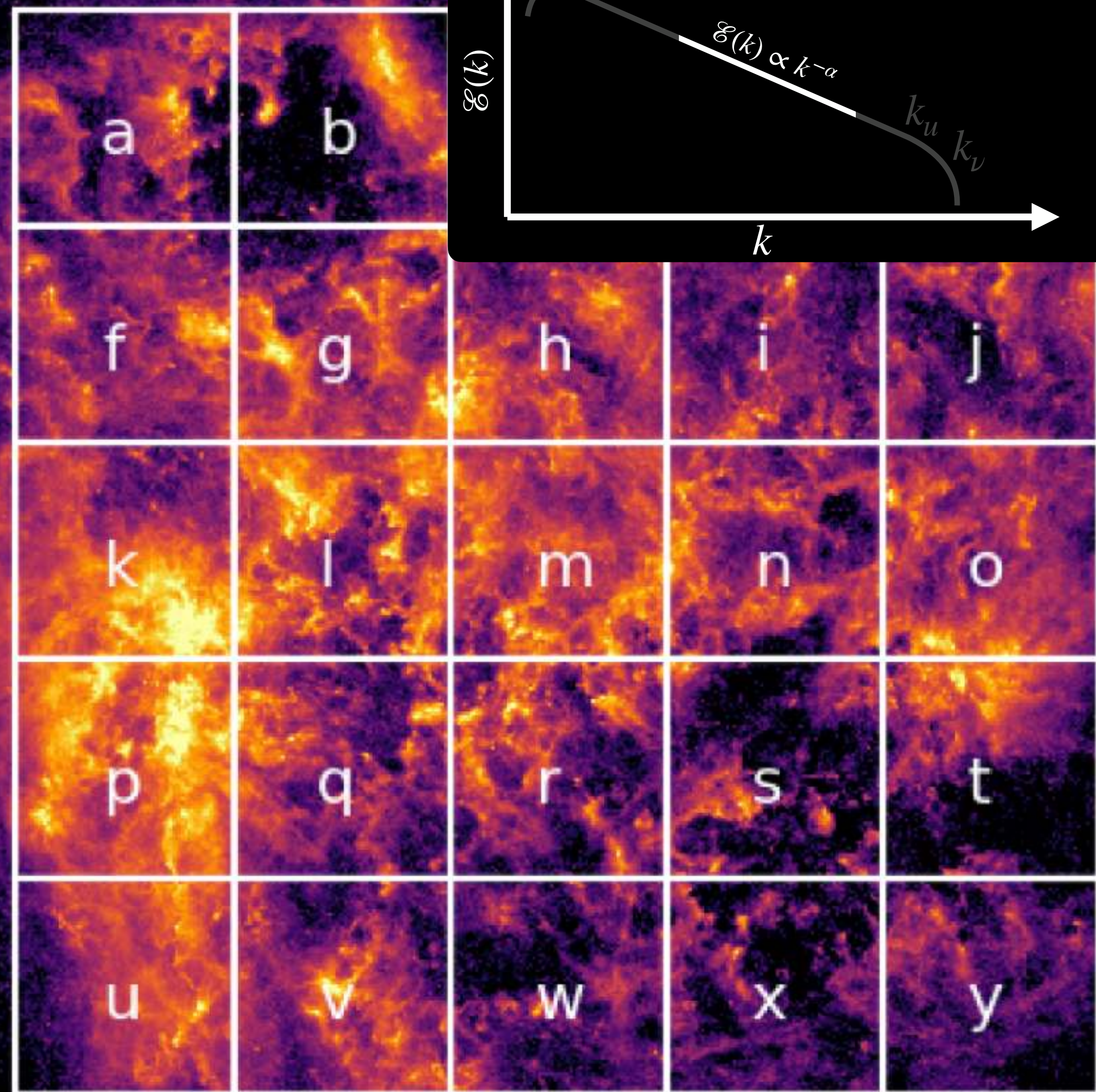
Turbulence



LMC: $500\mu\text{m}$, *Herschel* (processing Gordon et al. 2014)

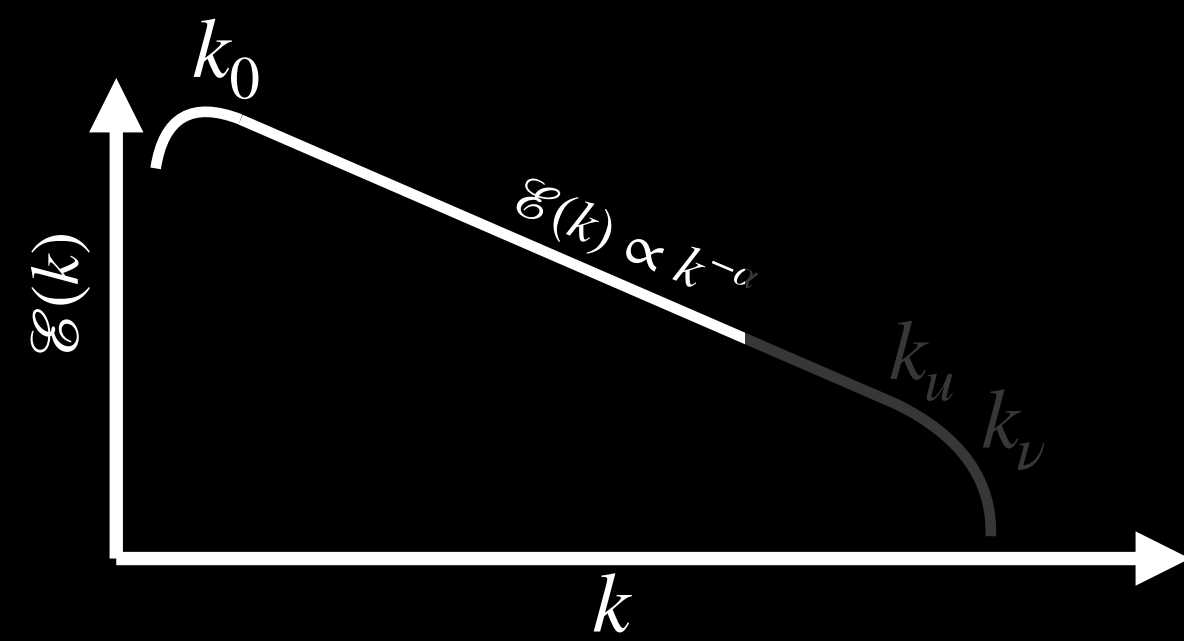


1 kpc



Colman, et al., 2022, MNRAS

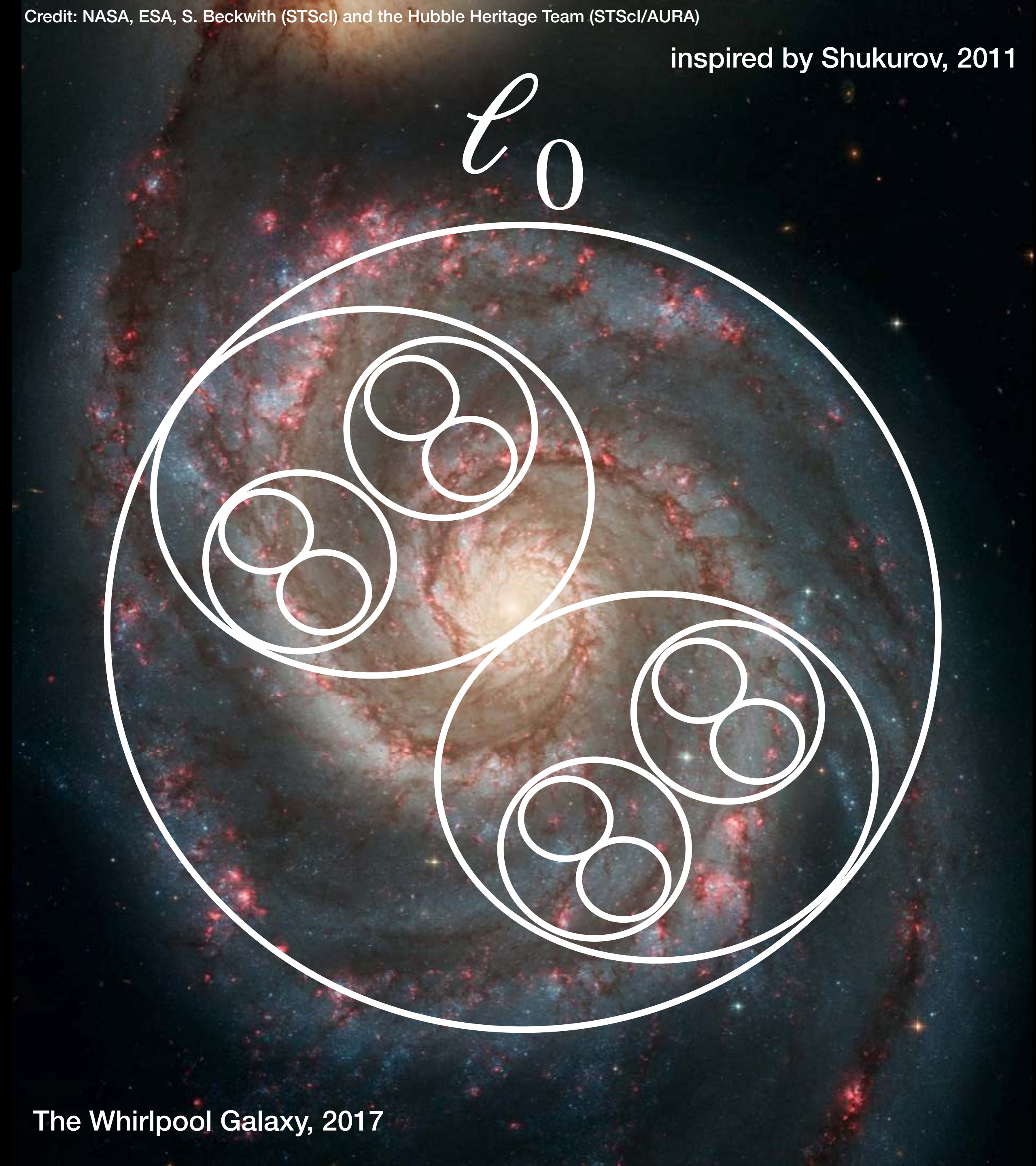
Turbulence Cascade



deeper into
the cascade

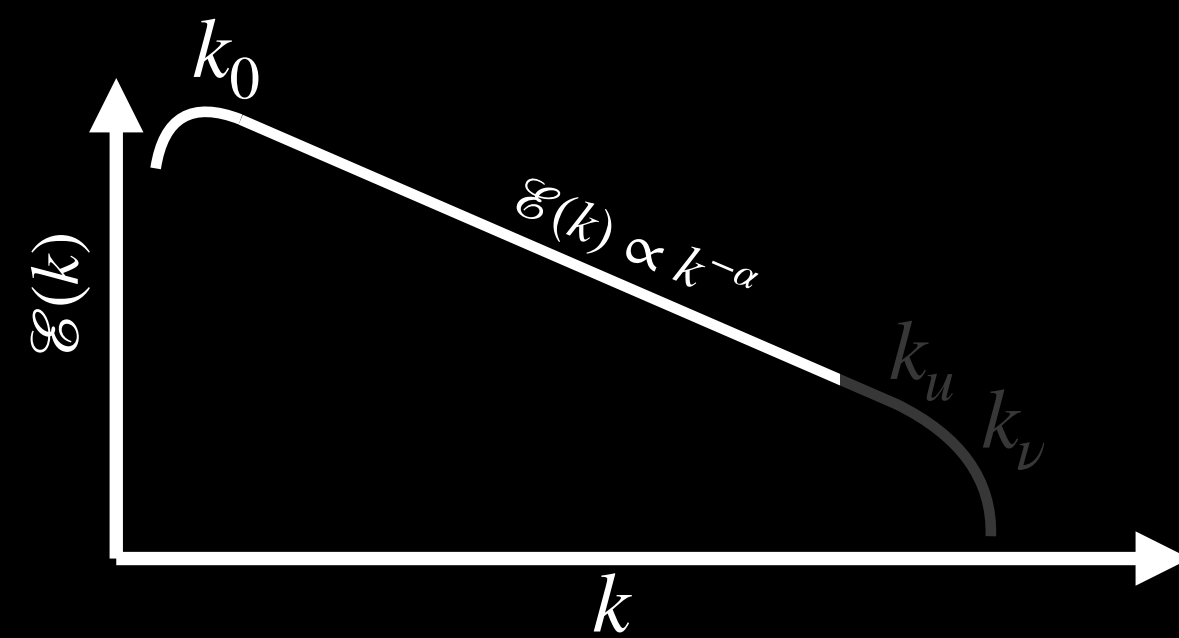
Credit: NASA, ESA, S. Beckwith (STScI) and the Hubble Heritage Team (STScI/AURA)

inspired by Shukurov, 2011



The Whirlpool Galaxy, 2017

Turbulence Cascade

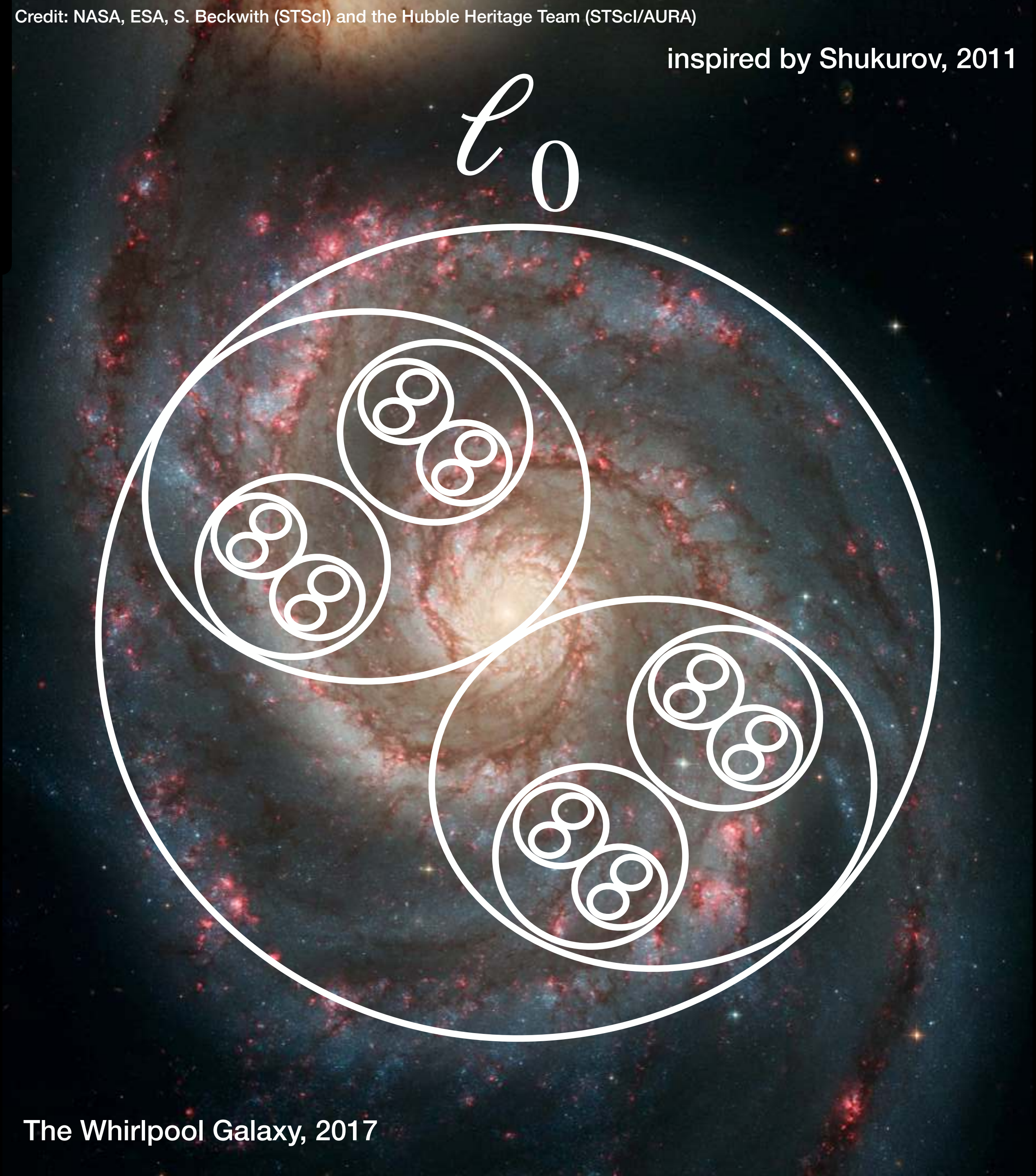
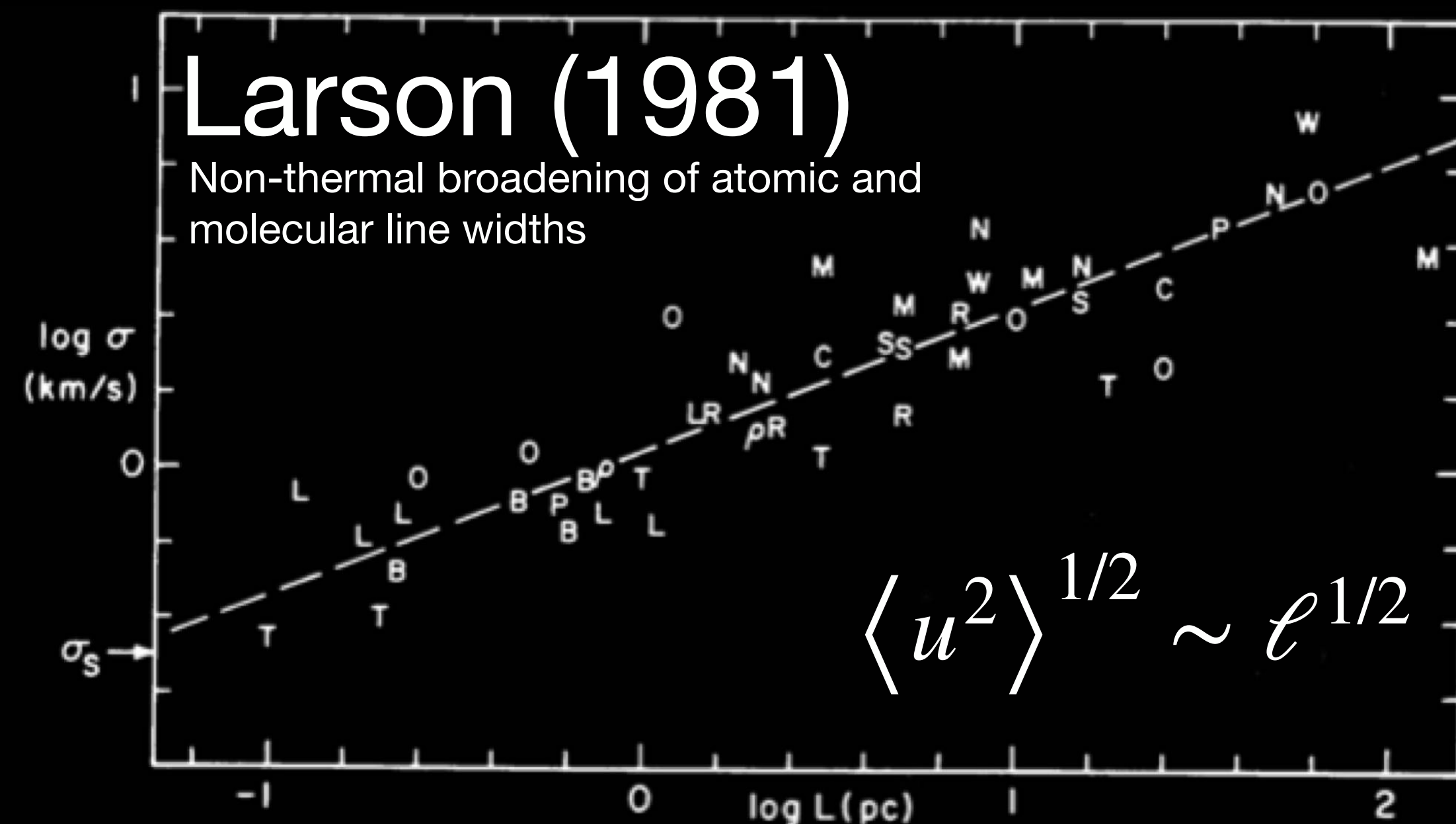


Credit: NASA, ESA, S. Beckwith (STScI) and the Hubble Heritage Team (STScI/AURA)

inspired by Shukurov, 2011

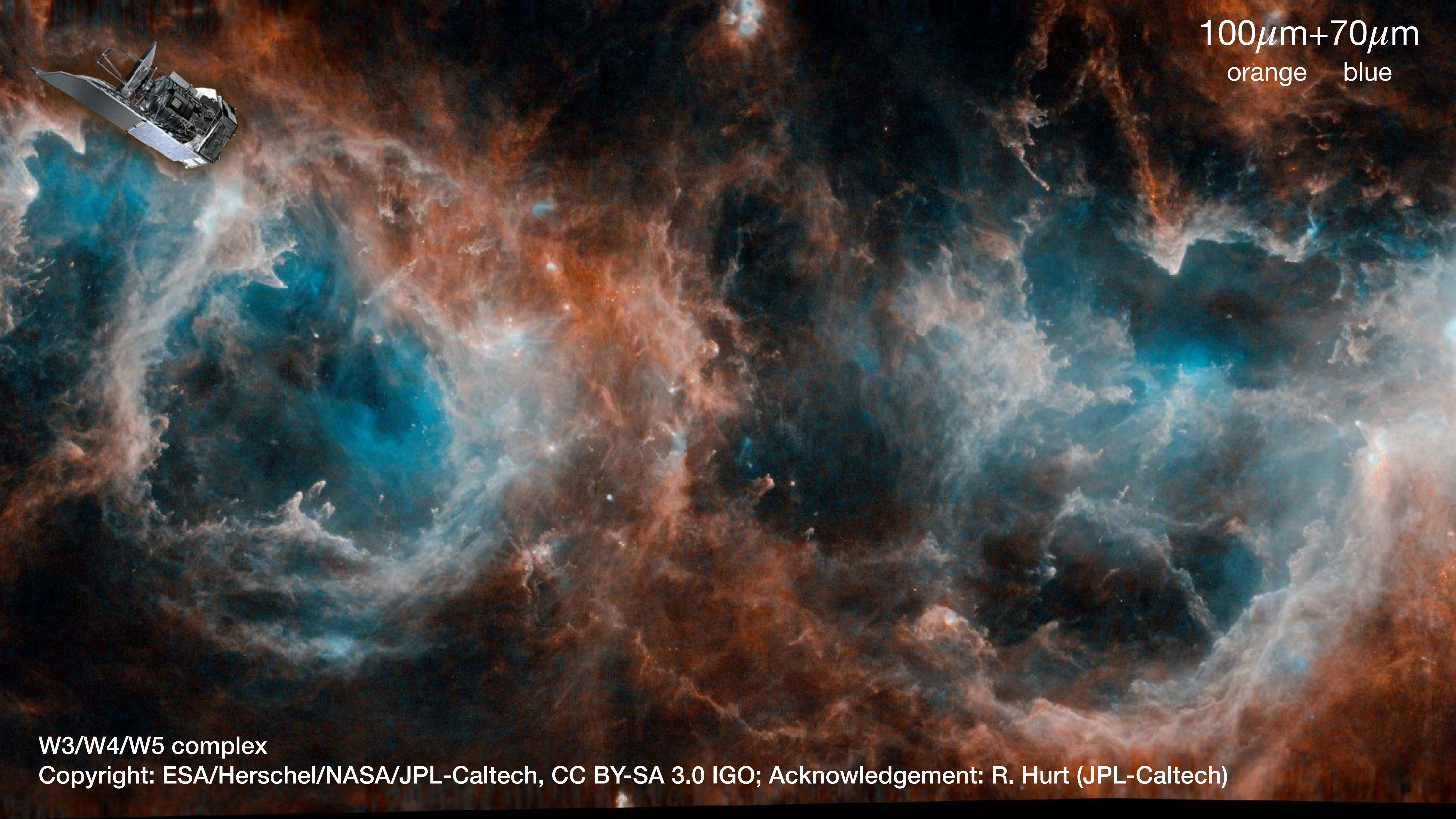
Reynolds number is shrinking

$$\text{Re} \sim \langle u^2 \rangle^{1/2} \ell / \nu \sim \ell^{3/2} / \nu$$



The Whirlpool Galaxy, 2017

$100\mu\text{m}+70\mu\text{m}$
orange blue



W3/W4/W5 complex

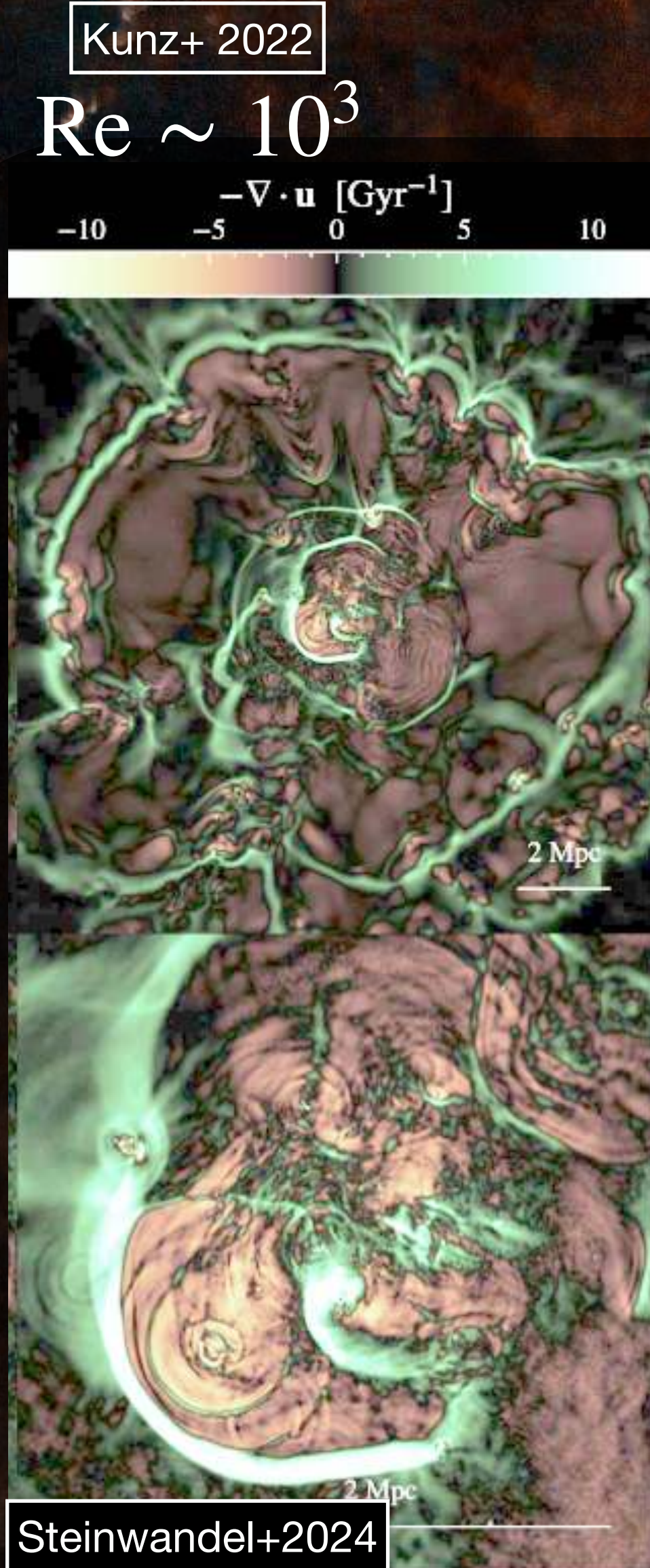
Copyright: ESA/Herschel/NASA/JPL-Caltech, CC BY-SA 3.0 IGO; Acknowledgement: R. Hurt (JPL-Caltech)

The cold ISM: A supersonic laboratory for interesting nonlinear physics

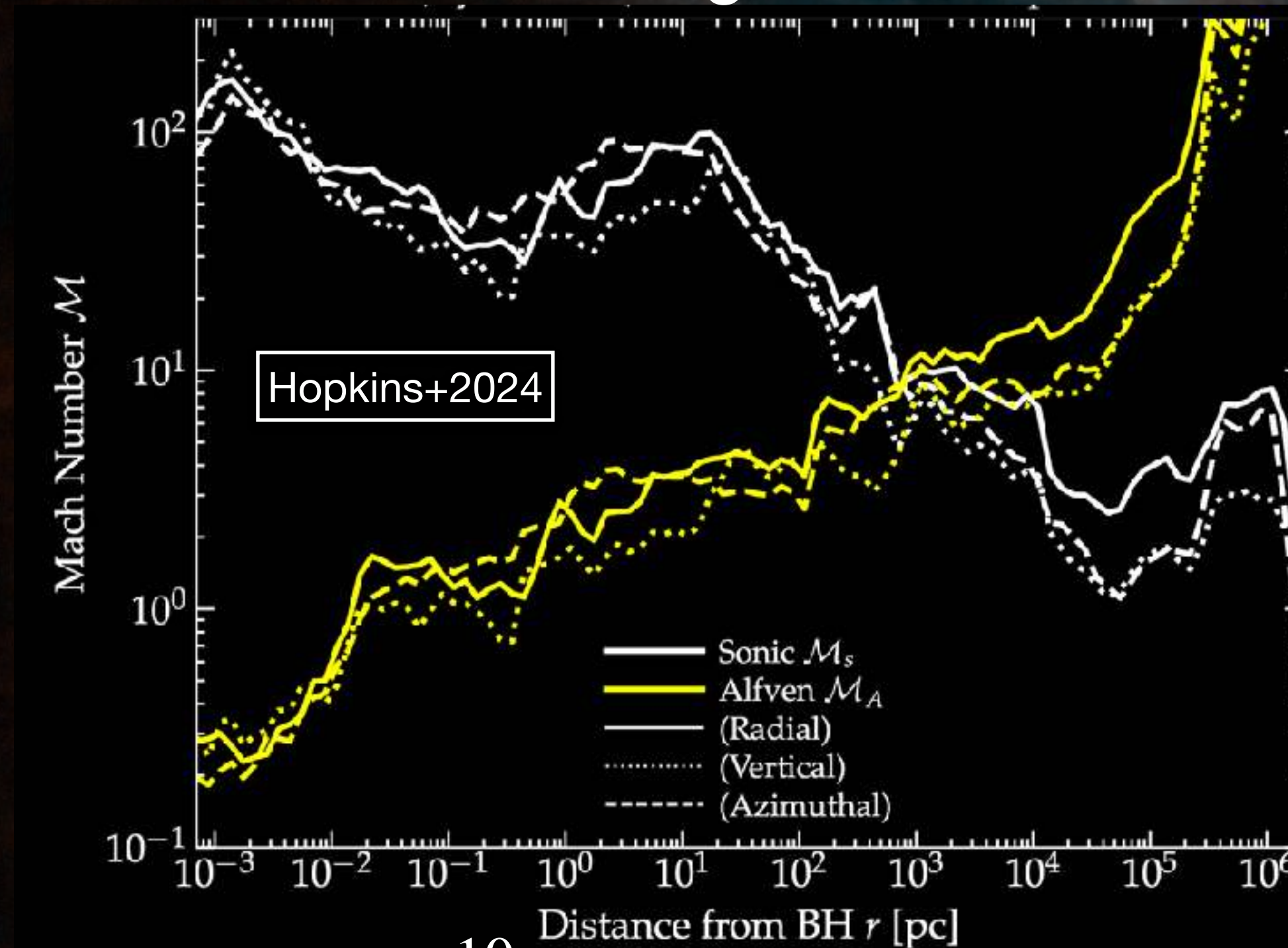
- $\sim 20\%$ MW ISM gas is molecular hydrogen, organised into MCs (low-volume filling $\sim 1\text{-}2\%$).
- cold, $T \sim 10\text{ K}$, c_s is low (and $t_{\text{cool}} \ll t_{\text{nl}}$), approximately isothermal
- $\sigma_v/c_s = \mathcal{M} \sim 20$, supersonic (compressible), $\text{Re} \sim 10^{6-9}$
- $L \sim 10\text{ pc} \implies T = L/\sigma_v \sim \mathcal{O}(\text{Myr})$
- $n \sim 10^3 - 10^{7+}$, huge density contrasts.
- weakly bounded (not virialised) by their own self-gravity
 $\alpha_{\text{vir}} = 2 |E_{\text{kin}}| / |E_{\text{grav}}| > 2$.
- threaded by dynamically important B fields, Ohmic
 $\text{Rm} \sim 10^{16}$.

Krumholz & McKee (2005); Federrath & Klessen (2012);
Tritsis+(2016); Xu & Lazarian (2016); Soler & Hennebelle (2017);
Squire & Hopkins (2017); Mocz & Burkart (2018), Burkart & Mocz
(2018); Heyer+(2012); Tritsis+(2018); Hu+(2019); Heyer+(2020);
Krumholz+(2020); Beattie & Federrath (2020); Beattie+(2020);
Körtgen & Soler (2020); Skalidis & Tassis (2020); Federrath+(2021);
Burkhart (2021); Barreto-Mota+(2021); Hopkins+(2022);
Beattie+(2022); Fielding+(2022); Sampson+ (2022); Kriel+(2022);
Galishnikova+(2022); Beattie+(2023)

Compressible turbulence is ubiquitous in many astrophysical plasmas



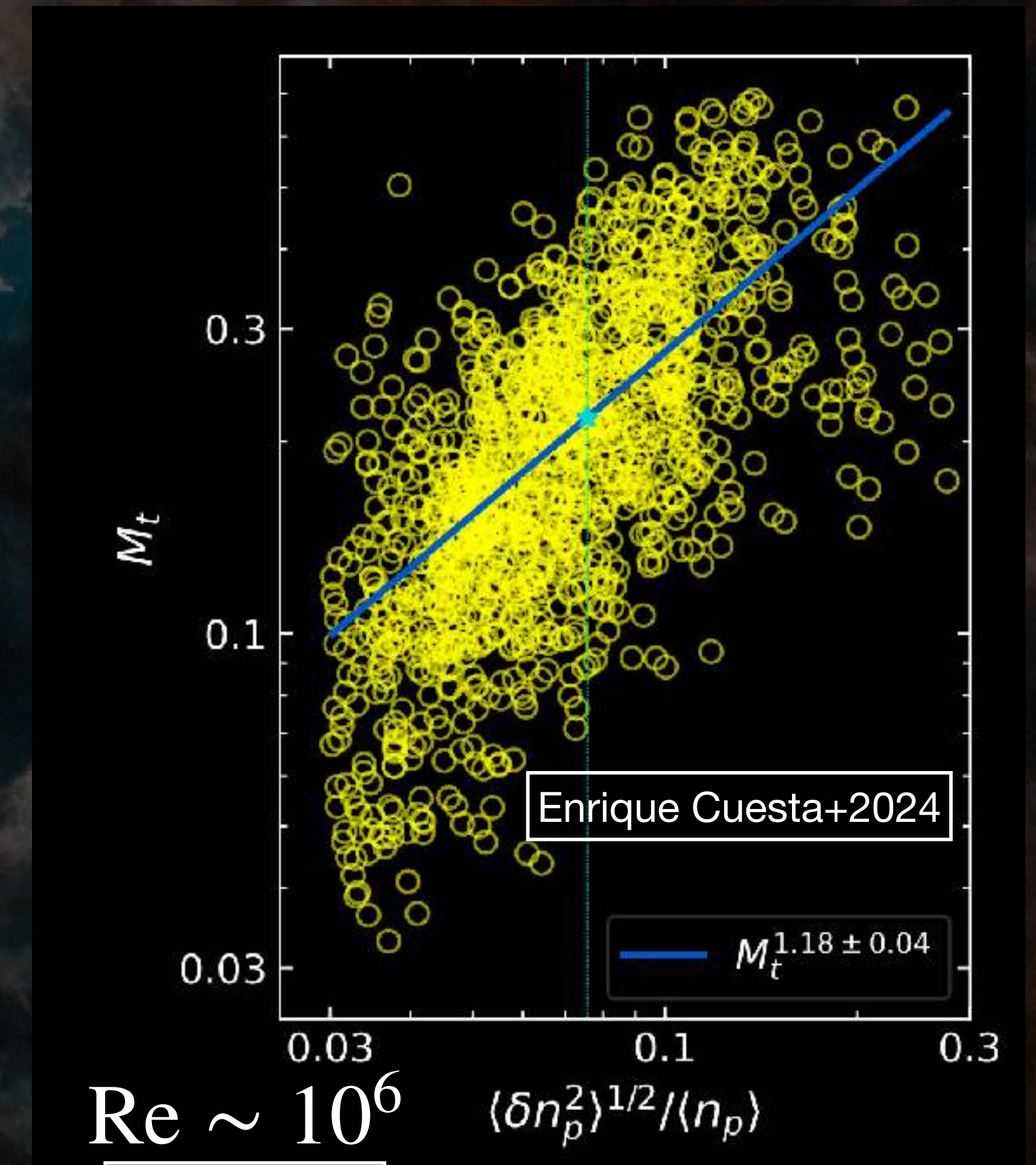
Accretion disks / galactic center



$Re \sim 10^{10}$

Rincon 2016

Solar wind close to the Sun



$Re \sim 10^6$

Wrench+2024

W3/W4/W5 MCs and star forming region complex.

ESA/Herschel/NASA/JPL-Caltech CC BY-SA 3.0 IGO; Acknowledgement: R. Hurt (JPL-Caltech).

10,080³ magnetized supersonic turbulence simulation

Beattie, Federrath, Klessen, Cielo &
Bhattacharjee

1. What is the scaling of the energy cascade in compressible MHD turbulence?
2. How are the characteristic scales organized in the compressible interstellar medium turbulence?
3. What are the saturation physics of the turbulent dynamo?

PI of a three total 200 million core-hour projects on superMUC-NG

ILES of compressible MHD turbulence

Turbulence: $\sigma_V/c_s \approx 4$, $\ell_0 = L/2$

Magnetic fields: $b_{\text{turb}}/B_0 = \infty$, $b_{\text{turb}}/B_0 = 1$

Three experiments for convergence tests:

- LOW-RES: 2,520³ (0.3Mcore-h, 8,640cores)
 - MID-RES: 5,040³ (4.0Mcore-h, 34,560cores)
 - **HIGH-RES: 10,080³ (80.0Mcore-h, 148,240cores)**
- ≈ 6 PB in data products $R_m \sim R_e \gtrsim 10^6$, $P_m \sim 1 - 2$

Broad Code details:

- Highly-modified version of finite volume code *FLASH*, second-order in space approximate Riemann solver with framework outlined in [Bouchut+2010](#), tested in *FLASH* in [Waagen+2011](#).
- Ideal (ILES) compressible non-helical, isothermal MHD turbulence with finite correlation time (OU process; [Federrath+2022](#)) driving in momentum.

DB: EXTREME_Turb_hdf5_plt_cnt_0100
Cycle: 118366 Time: 1.25

Pseudocolor
Var: mag - Speed

Max: 52.41
Min: 0.2508

Volume
Var: dens

Max: 319.1
Min: 0.002482

Volume
Var: dens

Max: 319.1
Min: 0.0009338

Volume
Var: dens

Max: 319.1
Min: 0.0009338

Volume
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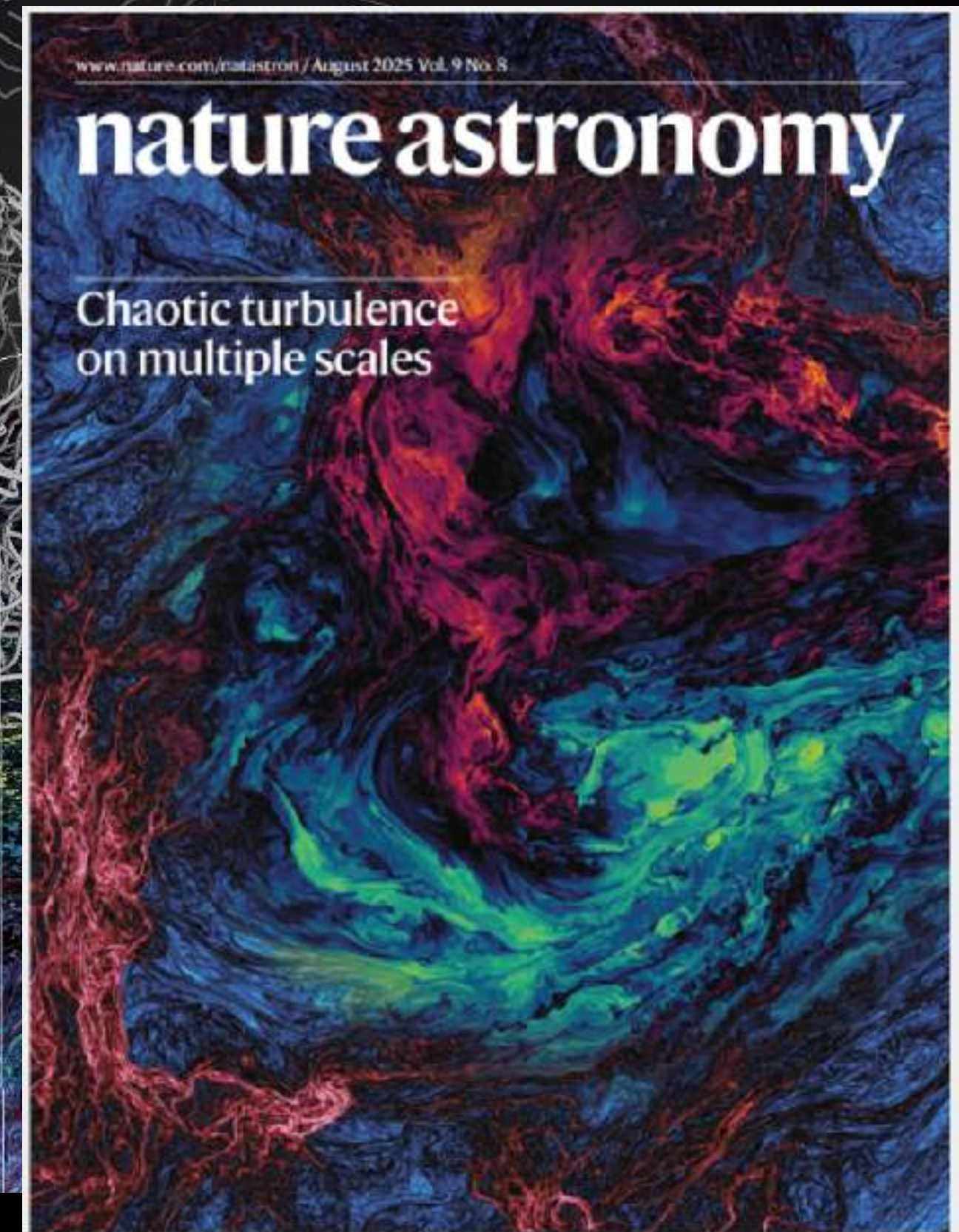
Volume
Var: dens

Max: 319.1
Min: 0.0009338

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Var: dens

Max: 319.1
Min: 0.0009338

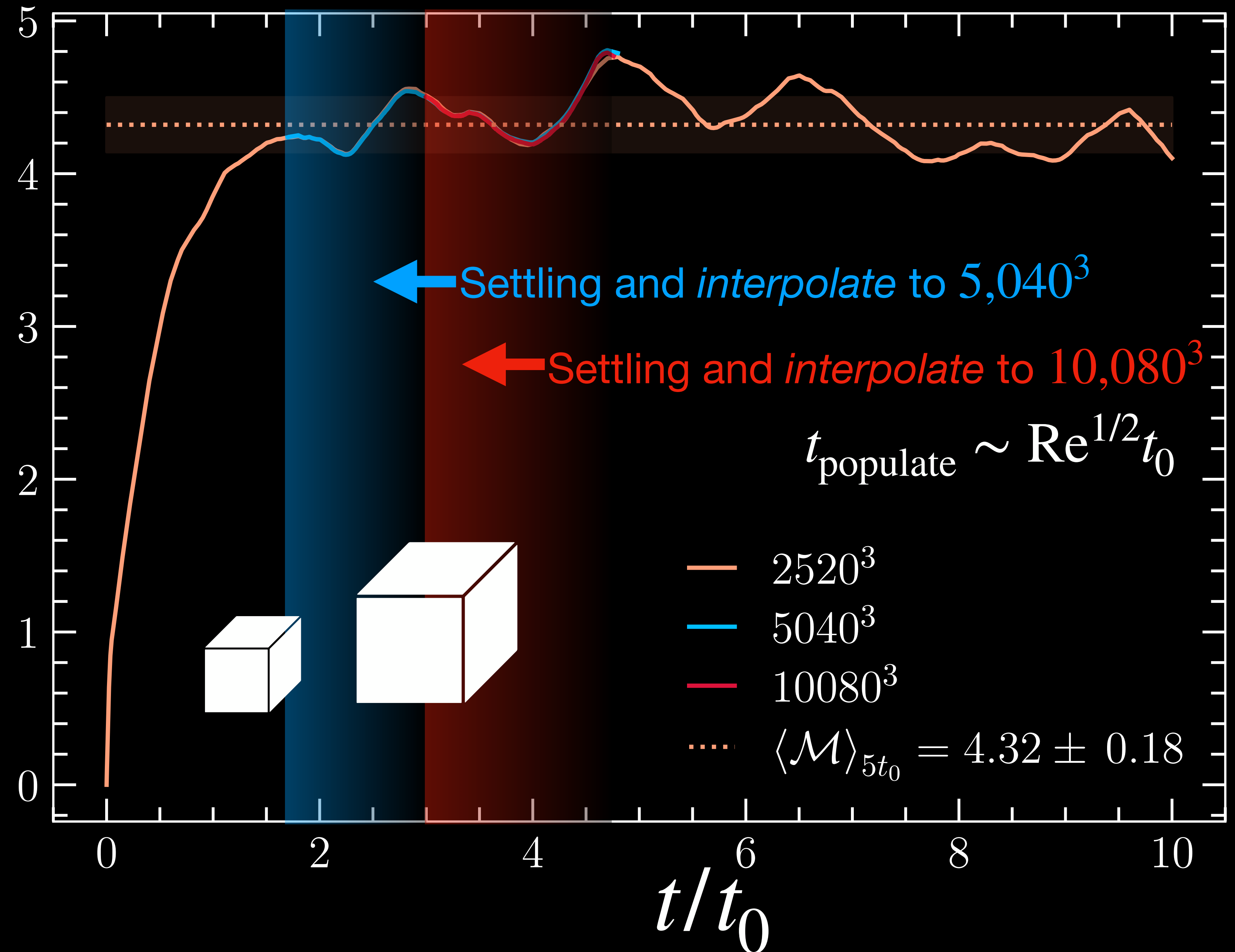
Volume
Var: dens



Volume integral Quantities

$$\mathcal{M} = \left\langle \frac{u^2}{c_s^2} \right\rangle_{\mathcal{V}}^{1/2}$$

Interpolation to
generate ICs for
successively higher
resolution
experiments

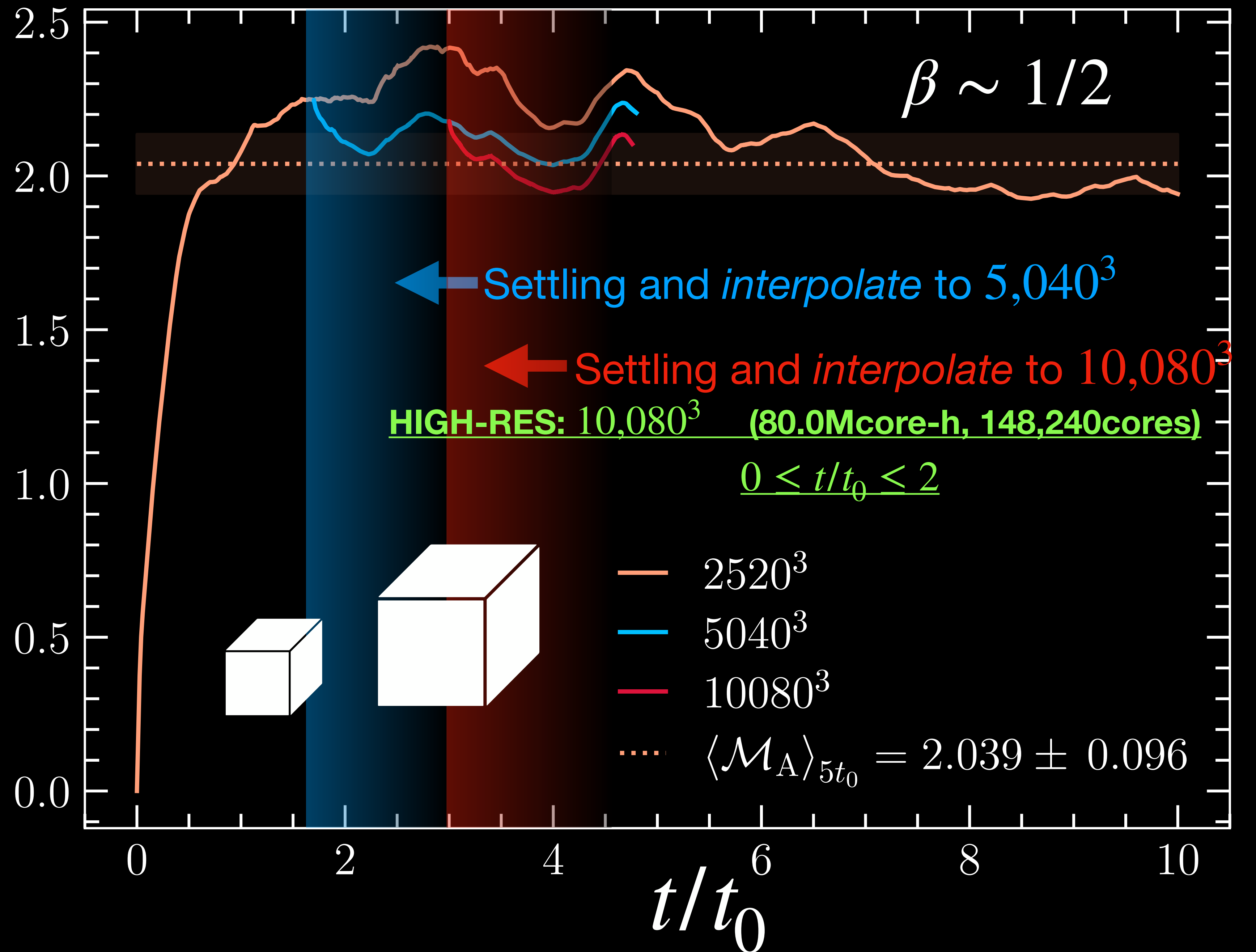


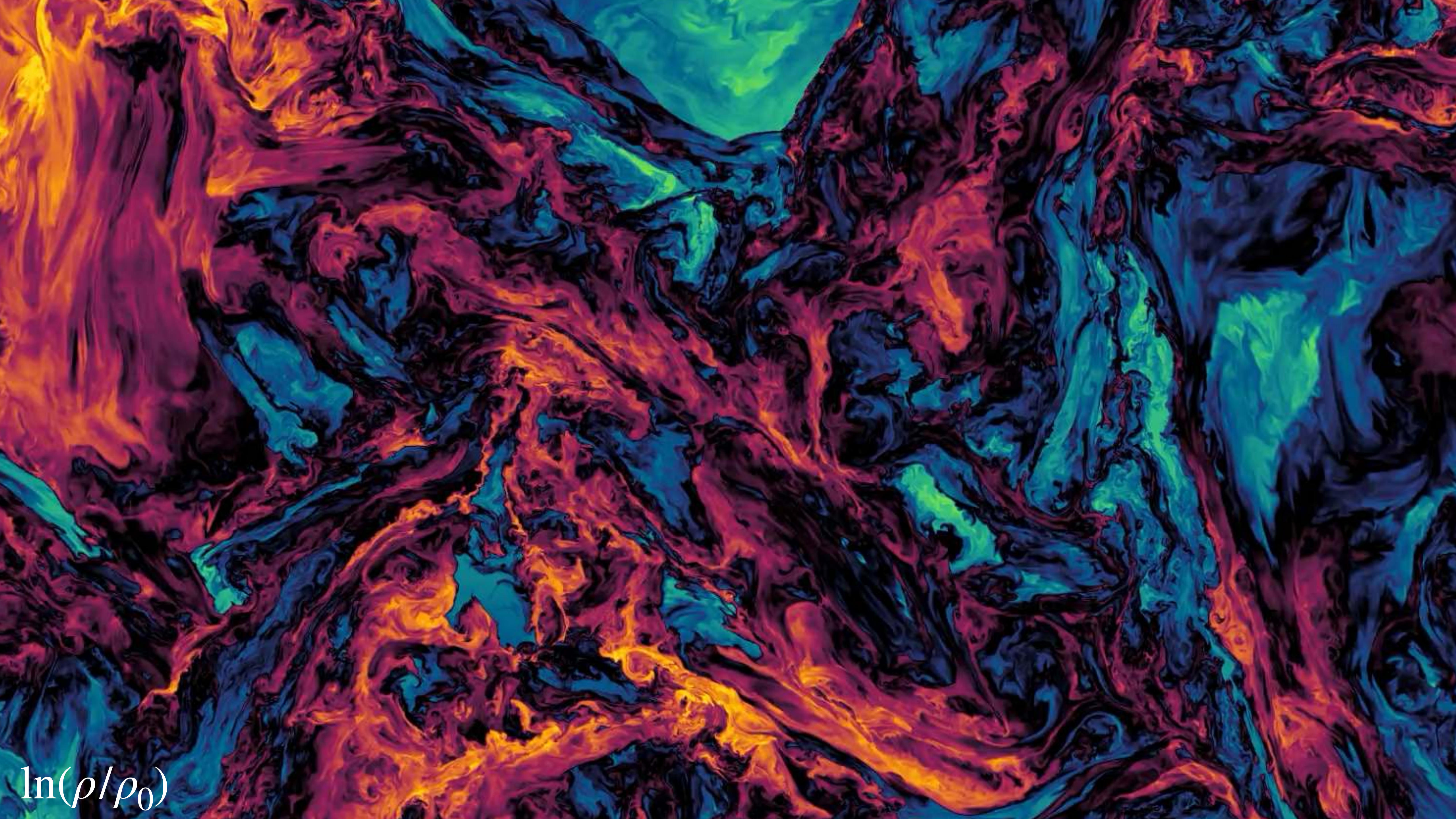
Volume integral Quantities

$$\mathcal{M}_A = \sqrt{\frac{E_{\text{kin}}}{E_{\text{mag}}}}$$

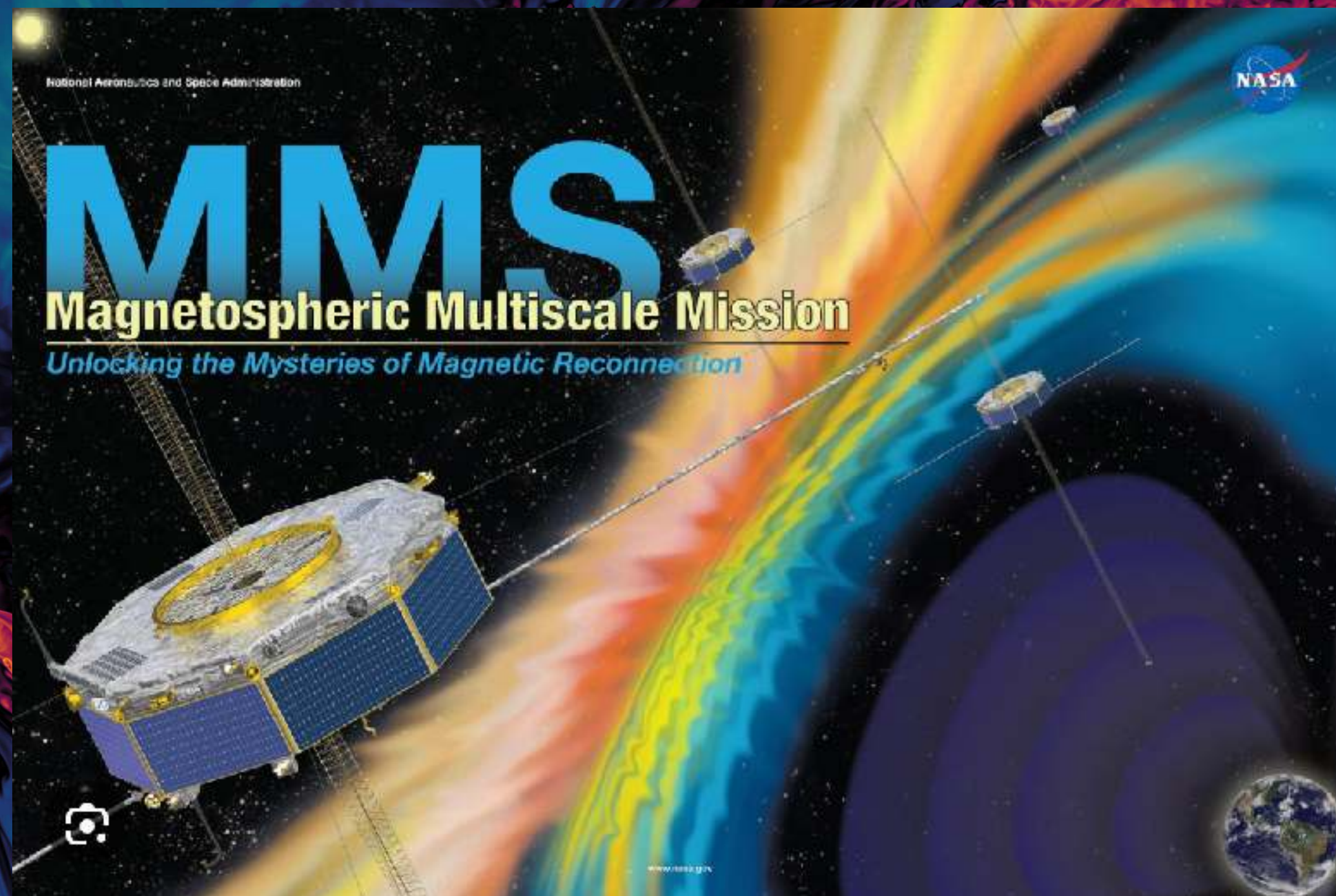
Interpolation to
generate ICs for
successively higher
resolution
experiments

$\mathcal{M}_A$

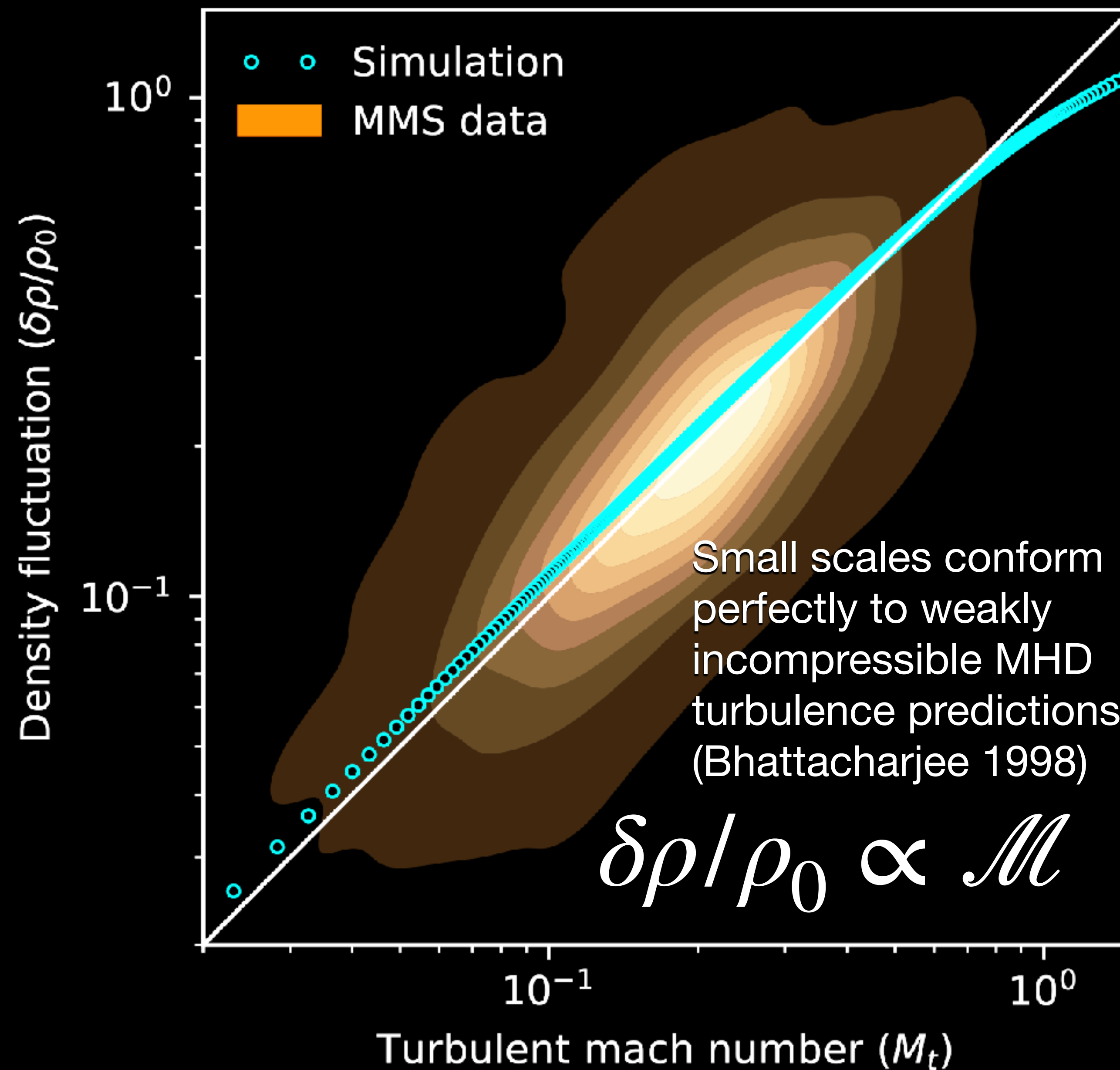




$\ln(\rho/\rho_0)$

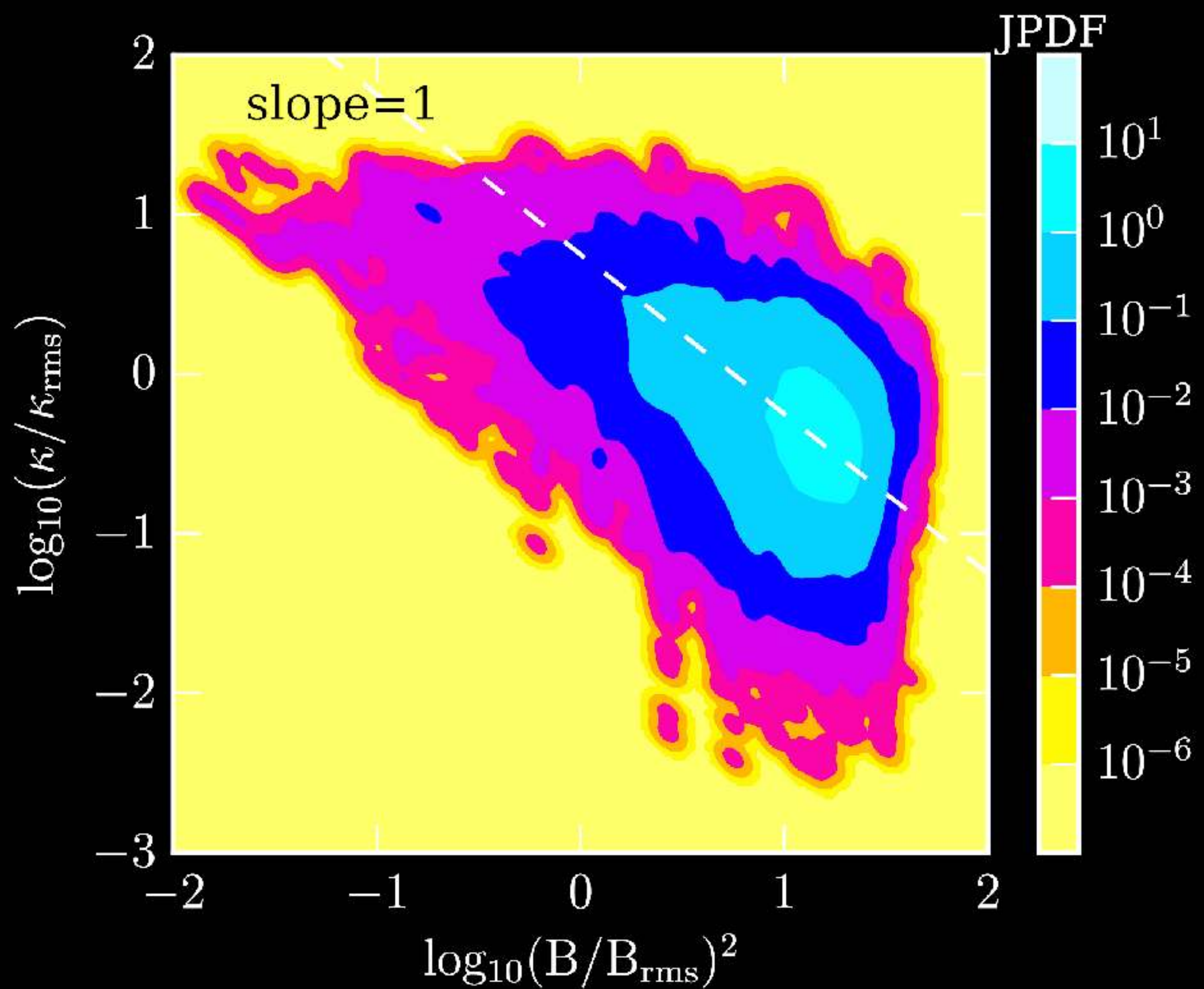


Bandyopadhyay, **Beattie** & Bhattacharjee (2025; ApJL)

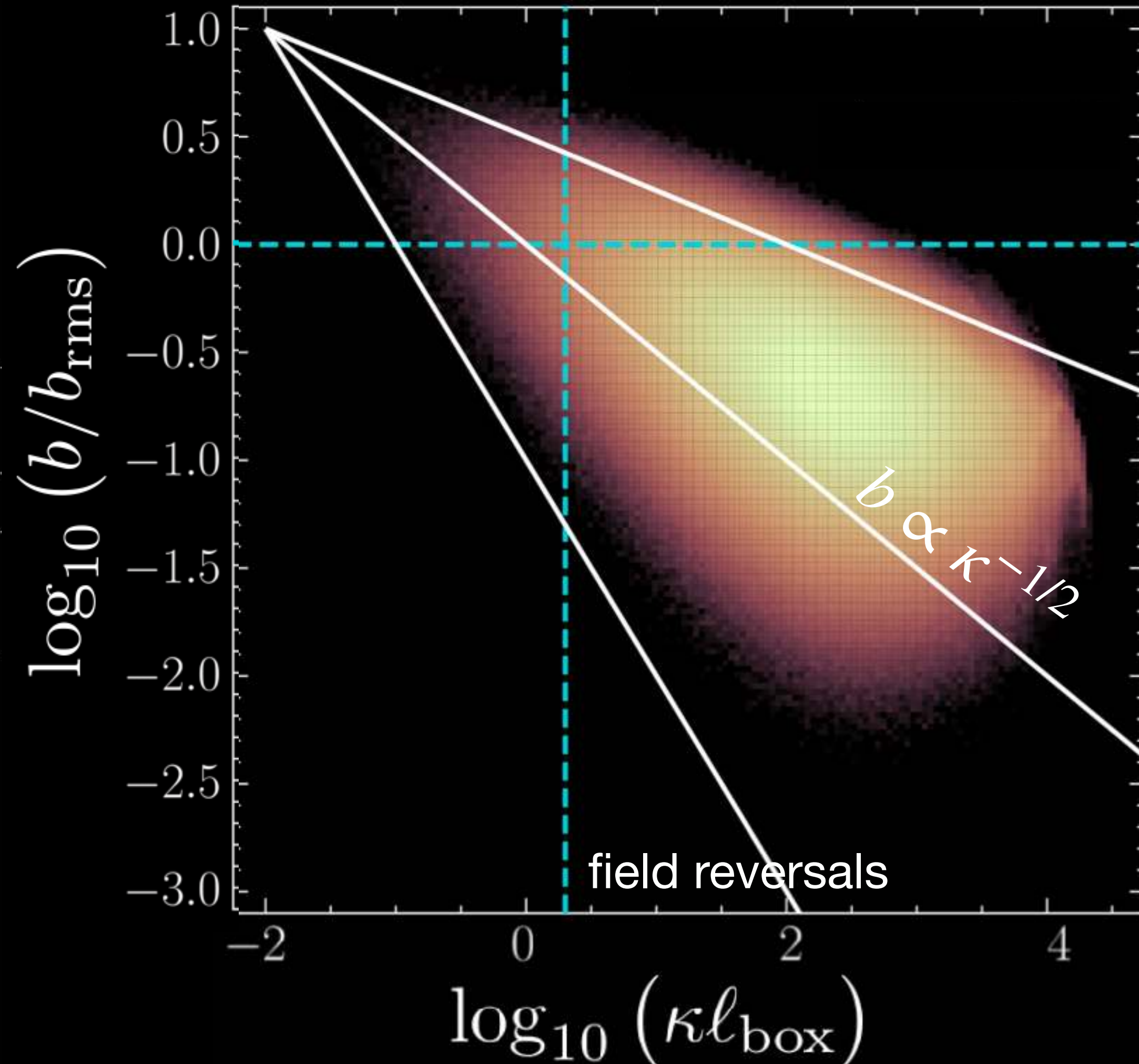


$\ln(\rho/\rho_0)$





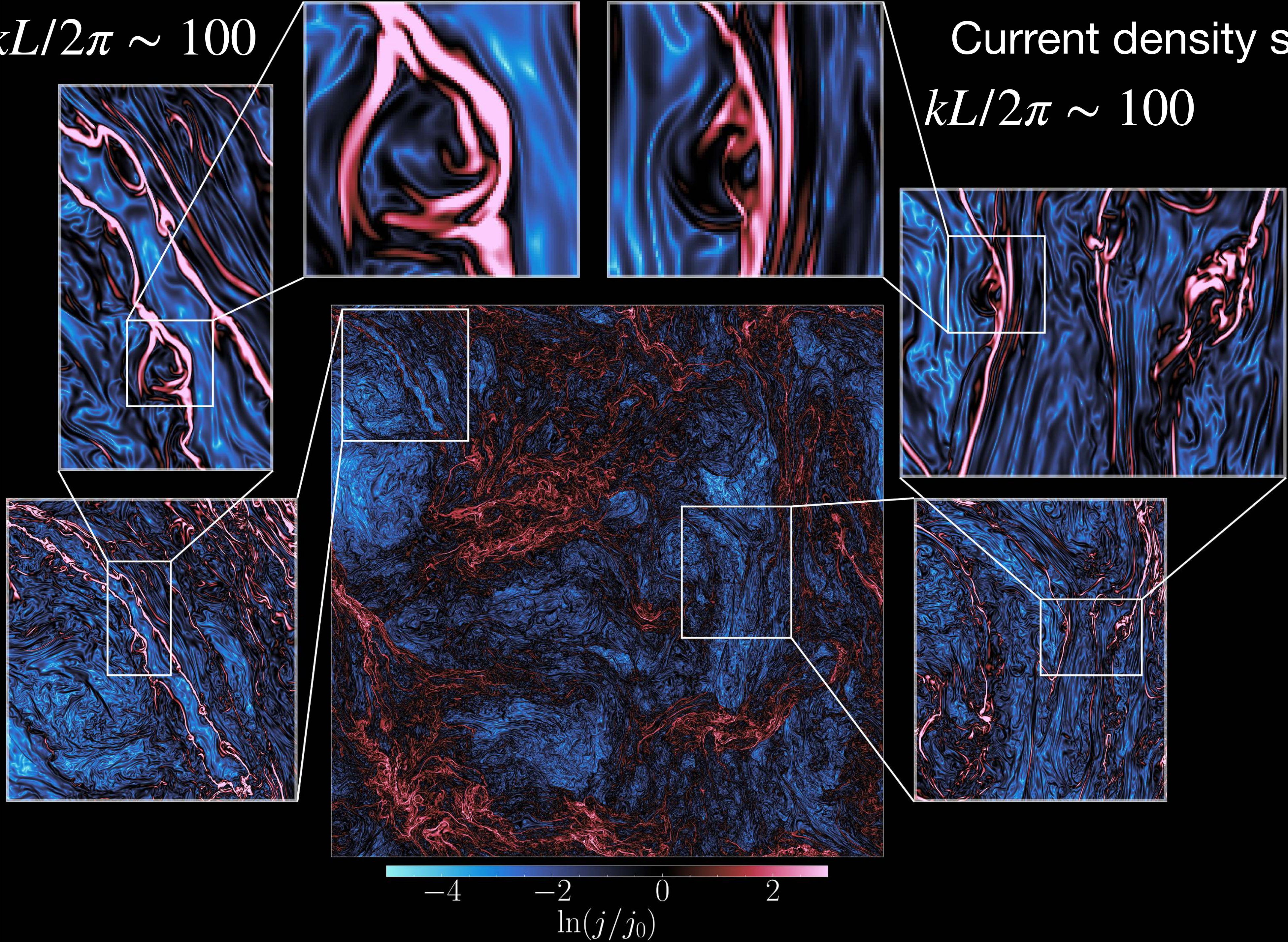
Bandyopadhyay+ (2024). *In situ measurement of curvature of magnetic field in turbulent space plasmas: a statistical study*



Follows exactly small-scale dynamo prediction
Schekoshihin+(2004)

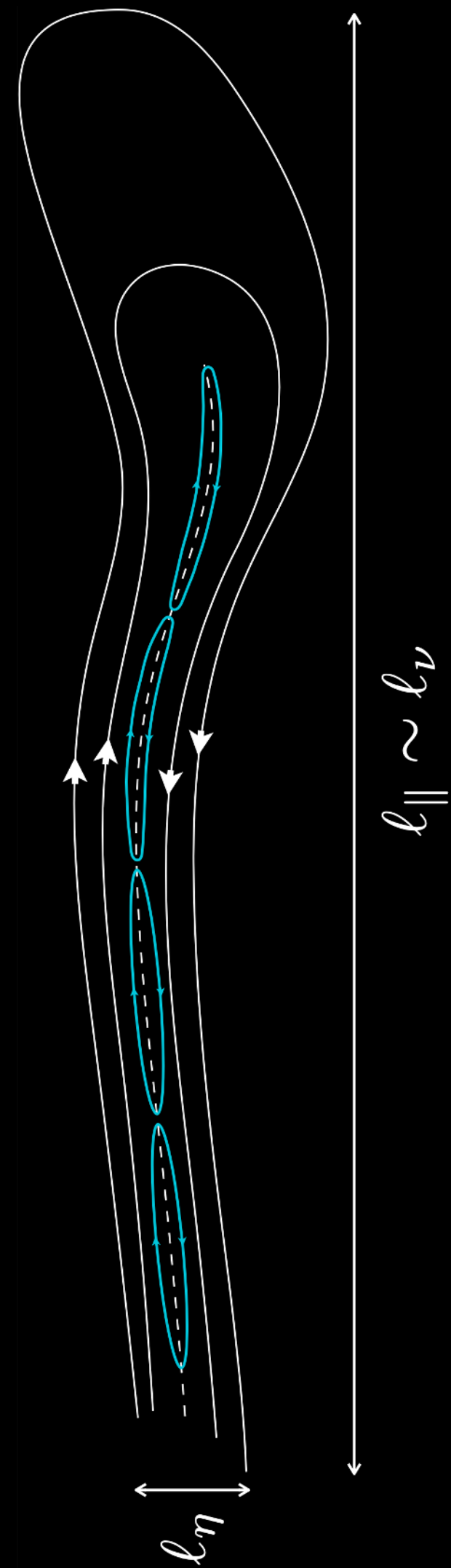
$kL/2\pi \sim 100$

Current density structures
 $kL/2\pi \sim 100$

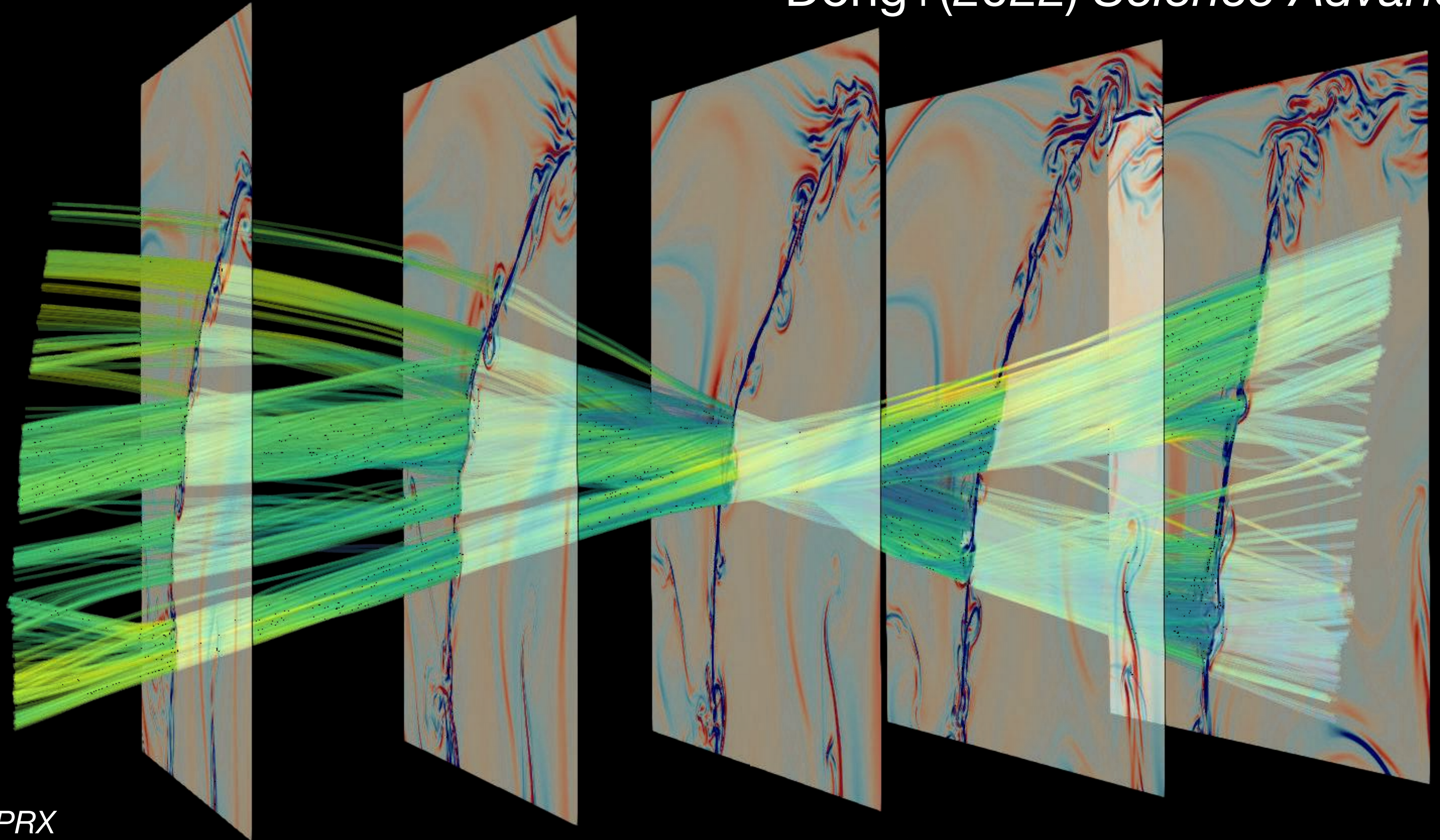


Turbulence meets fast reconnection

Dong+(2022) *Science Advances*



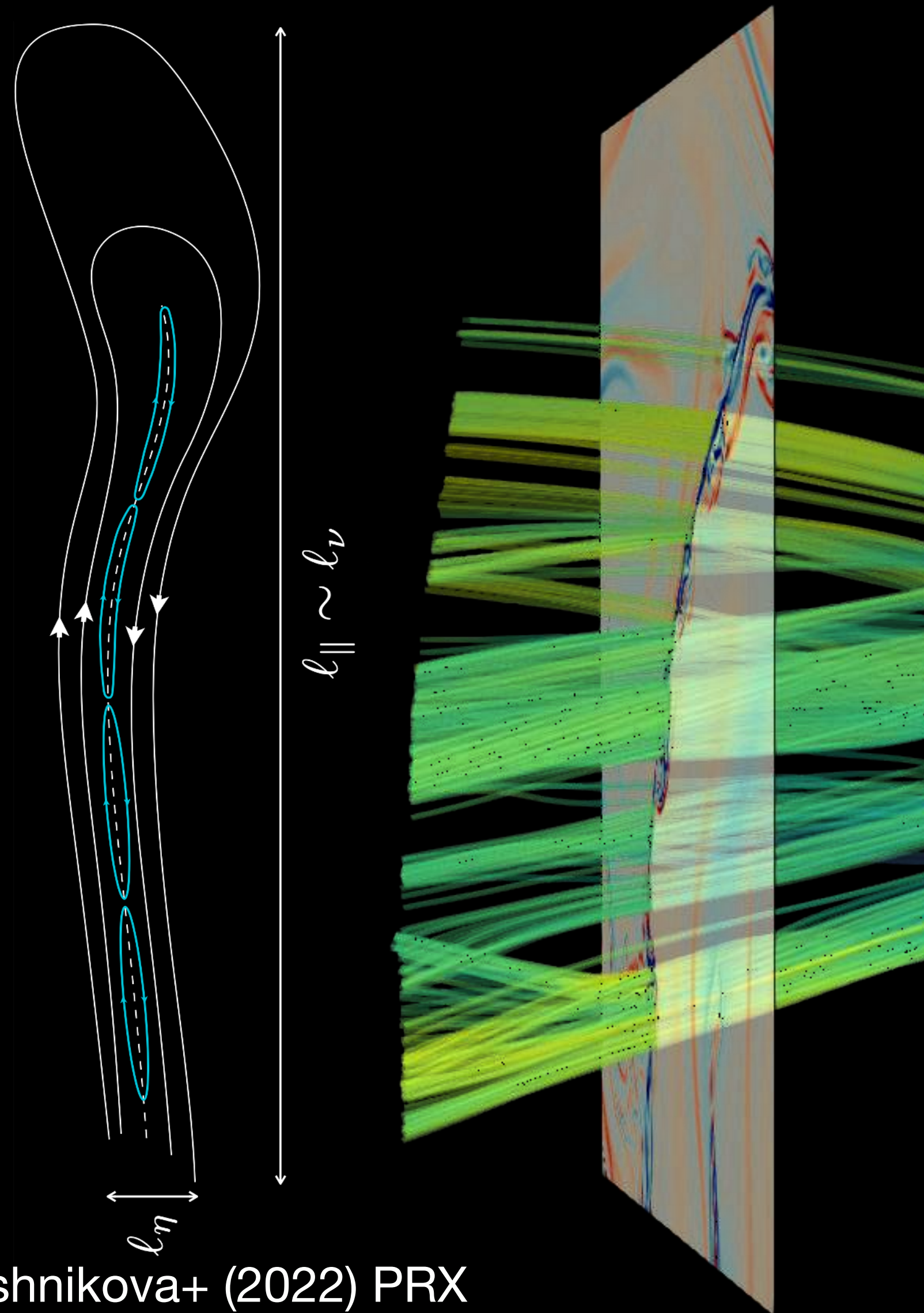
Galishnikova+ (2022) *PRX*



tearing instabilities growing inside a 3D current sheet

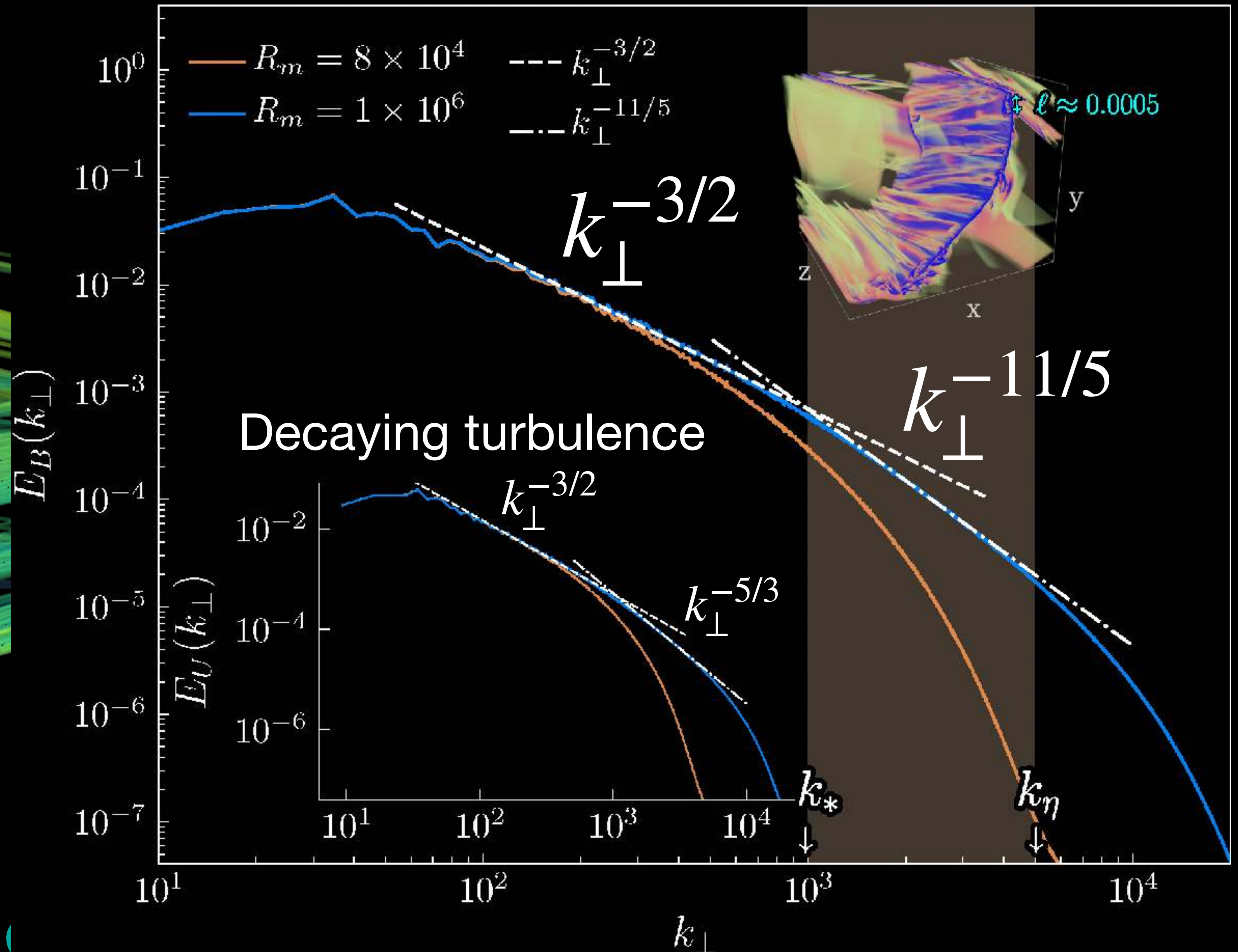
Turbulence meets fast reconnection

Dong+(2022) *Science Advances*

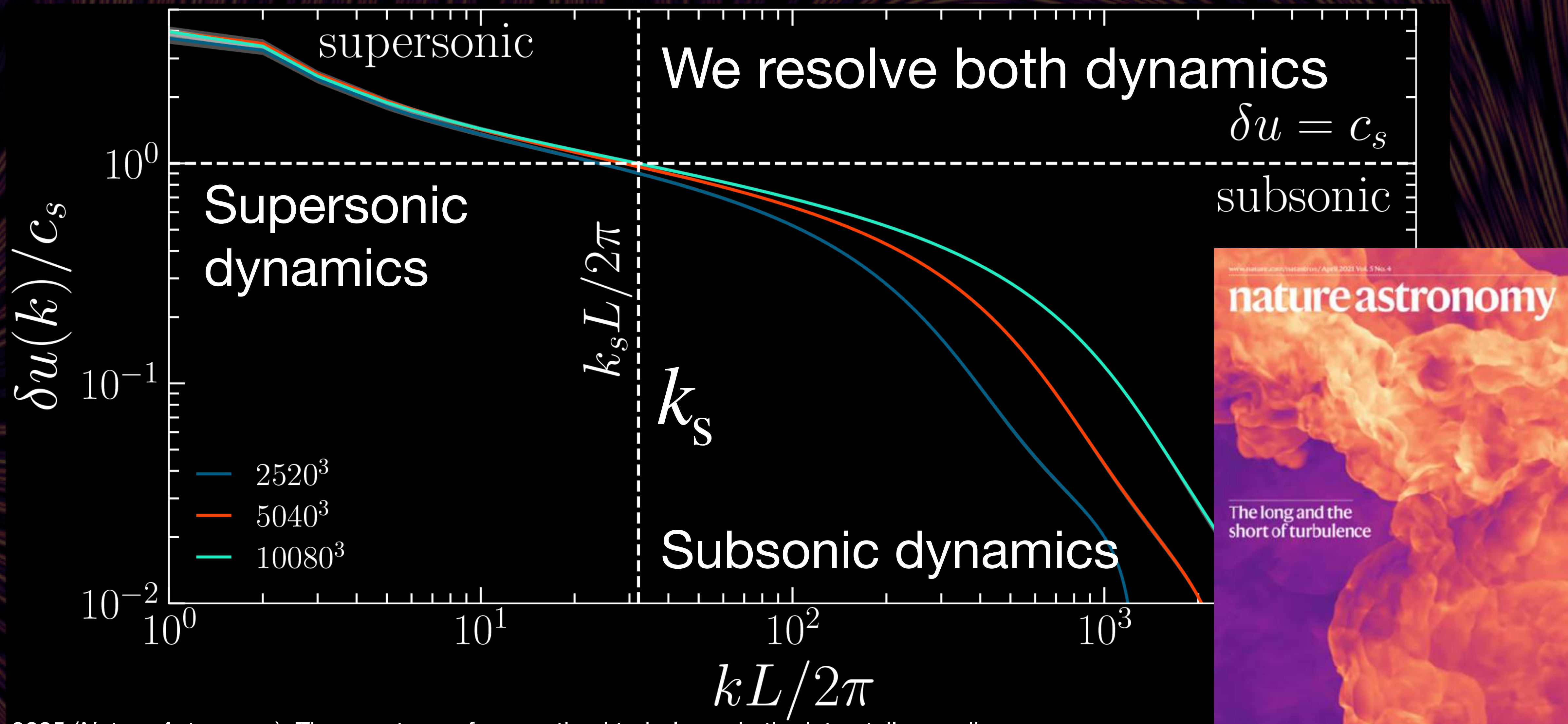


Galishnikova+ (2022) PRX

tearing instabilities



Two important scales — the sonic scale



Beattie+ 2025 (*Nature Astronomy*). The spectrum of magnetized turbulence in the interstellar medium

Magnetic reconnection in a turbulent plasma (visualisation by J. Beattie).

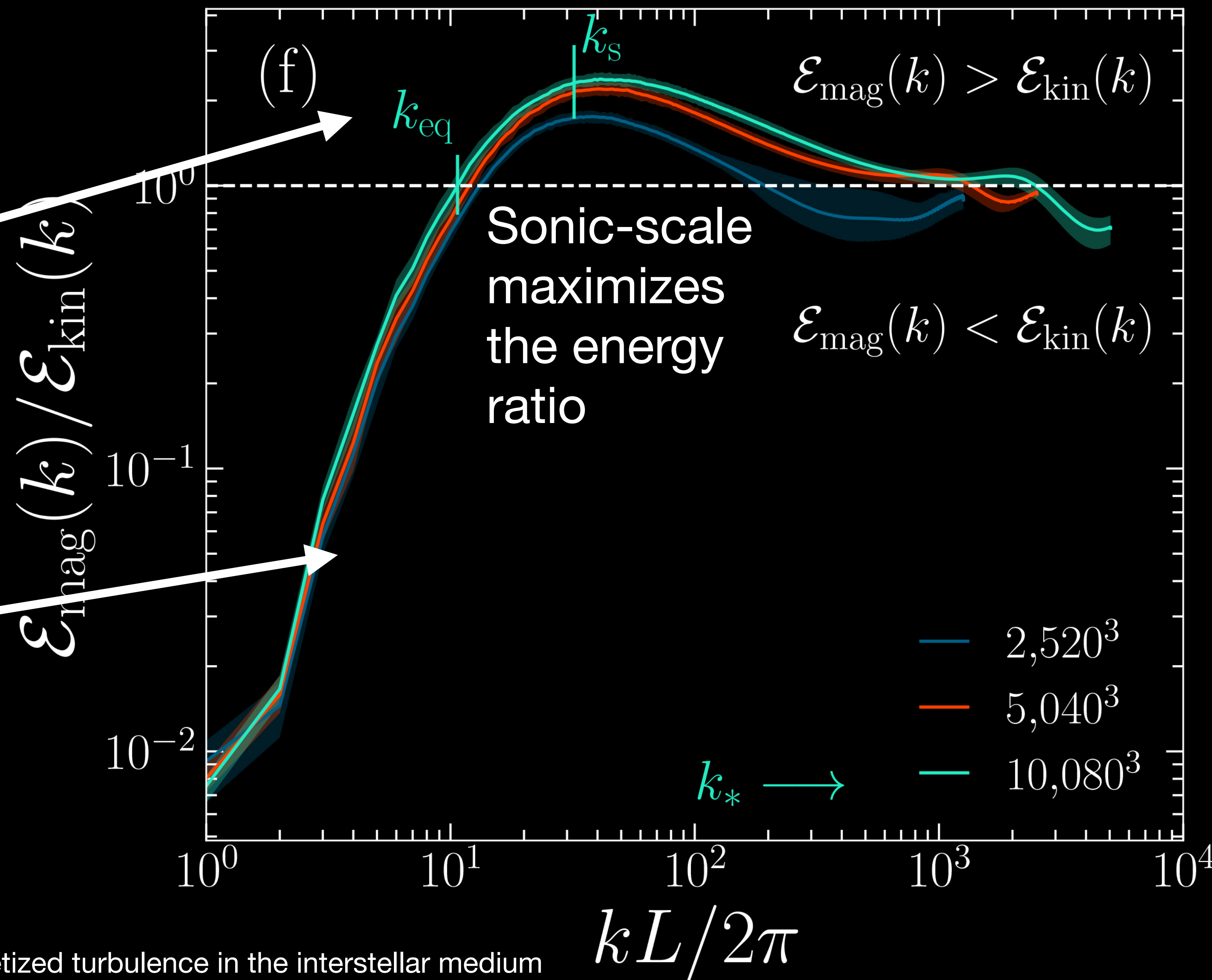
Two important scales — the Alfvén scale

sub-Alfvénic

k_{eq}

super-Alfvénic

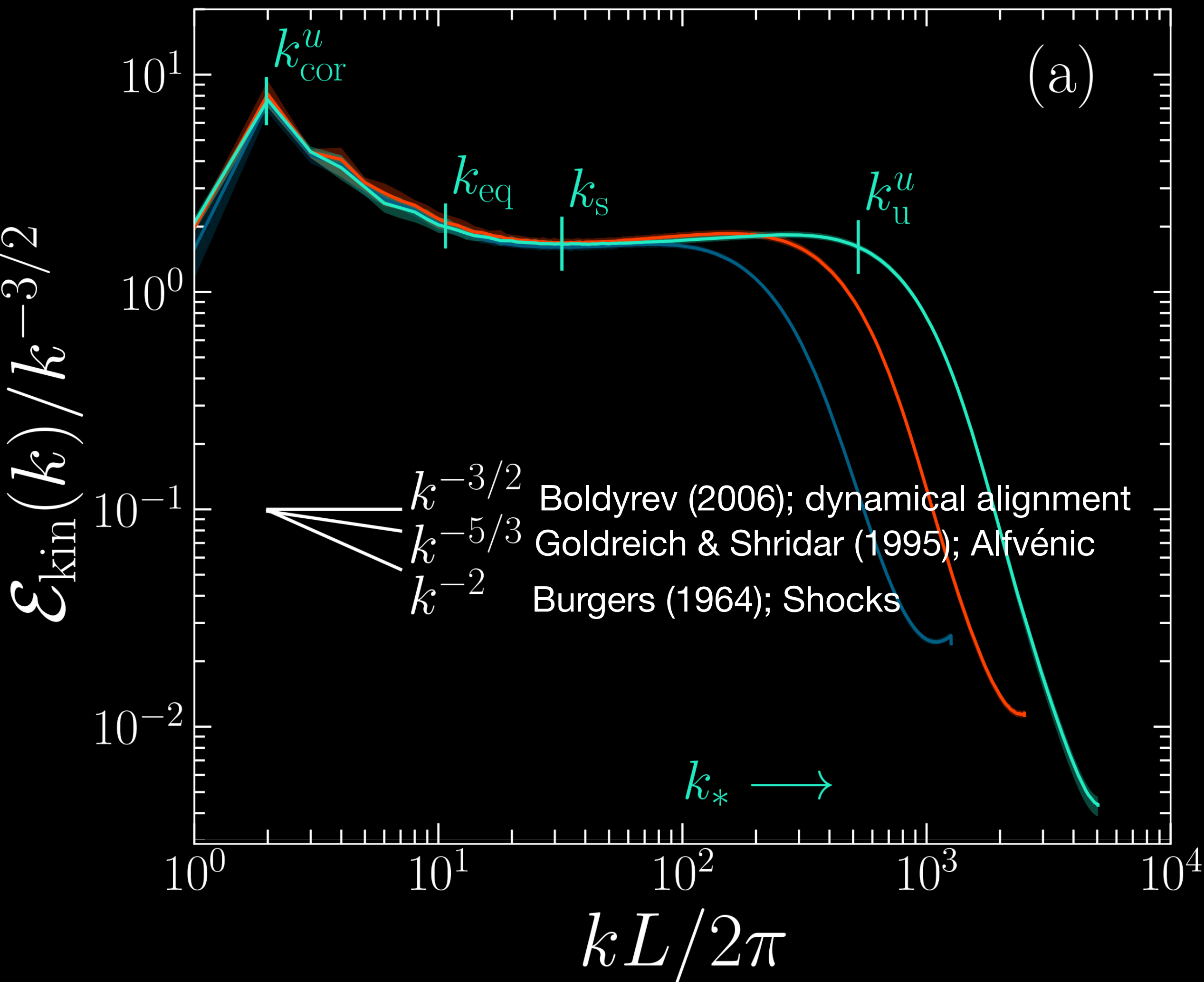
Multiple regimes
in a single domain!



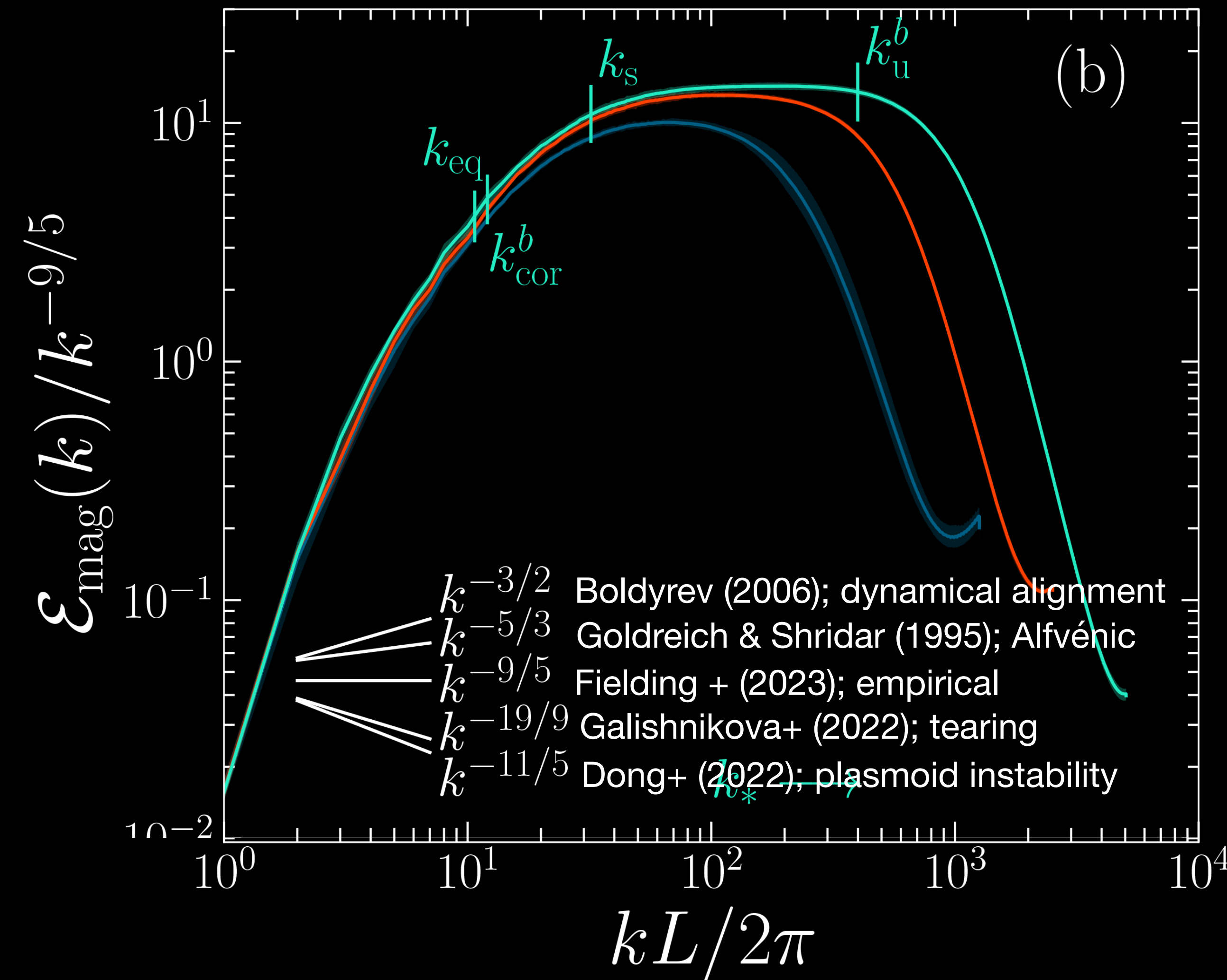
Beattie+ 2025 (*Nature Astronomy*). The spectrum of magnetized turbulence in the interstellar medium

Magnetic reconnection in a turbulent plasma (visualisation by J. Beattie).

Energy spectra: the disparate lives of u and b fluctuations

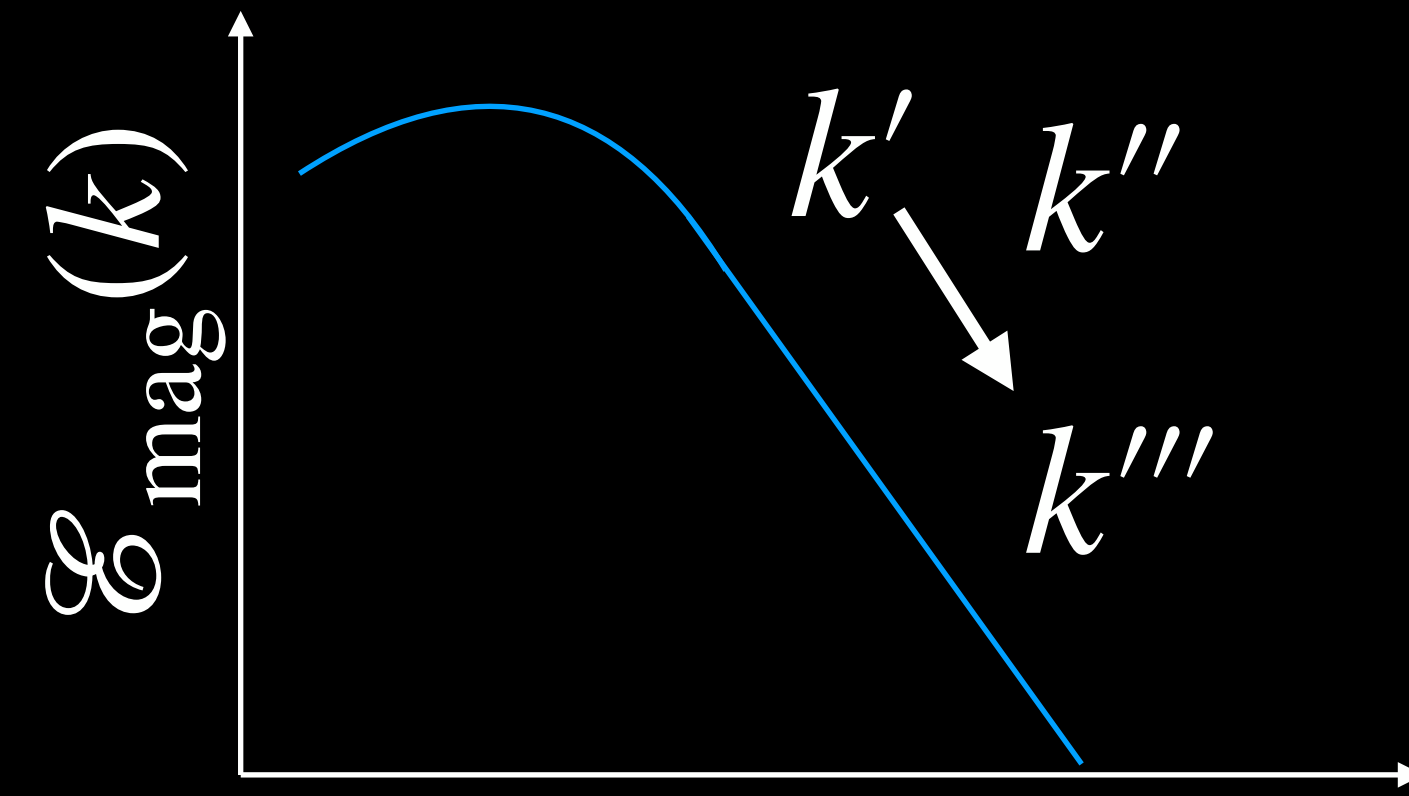
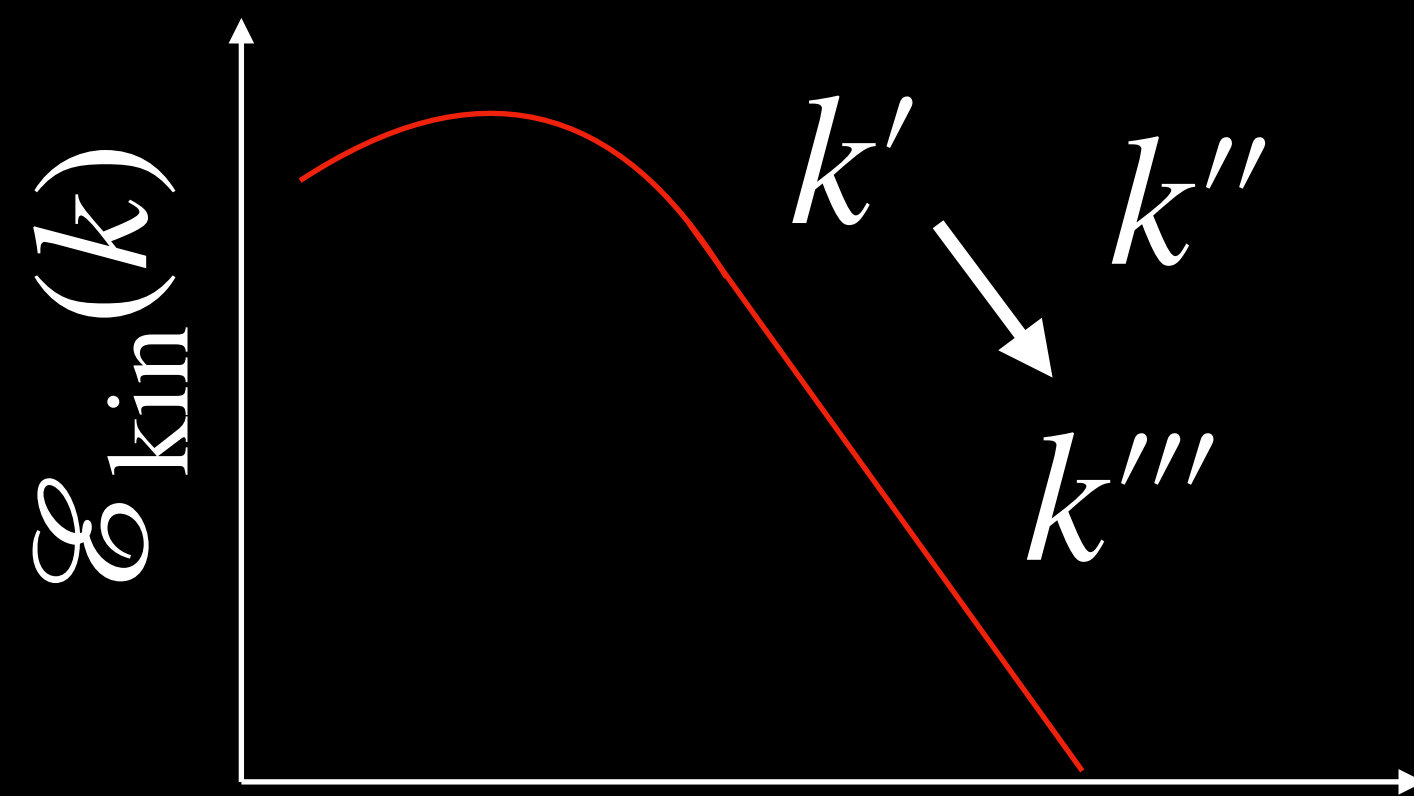


Two power laws



One power law, only
emerging at very high Rm

Energy spectra: the disparate lives of u and b fluctuations



$$\mathbf{w} = \sqrt{\rho} \mathbf{u}$$

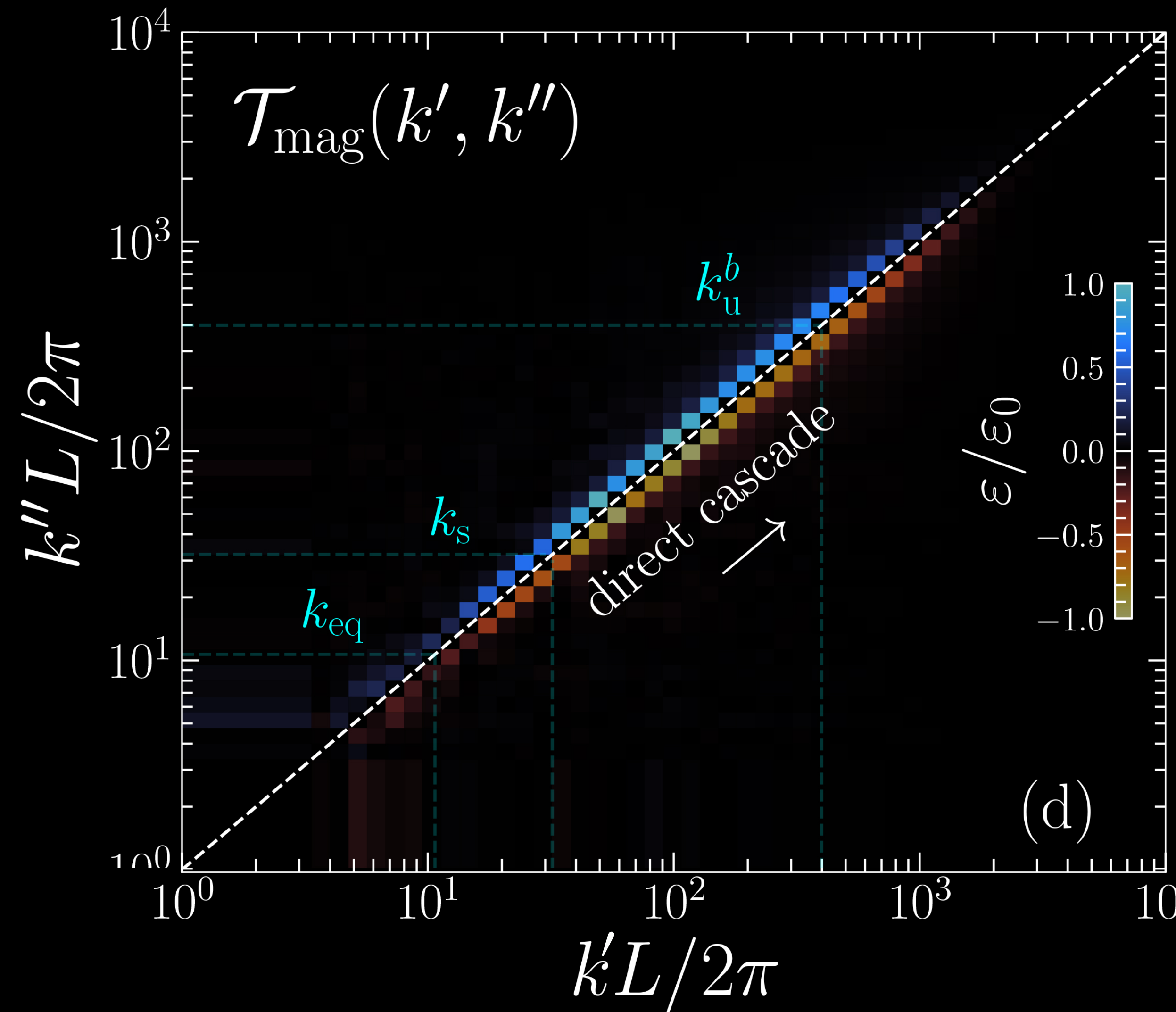
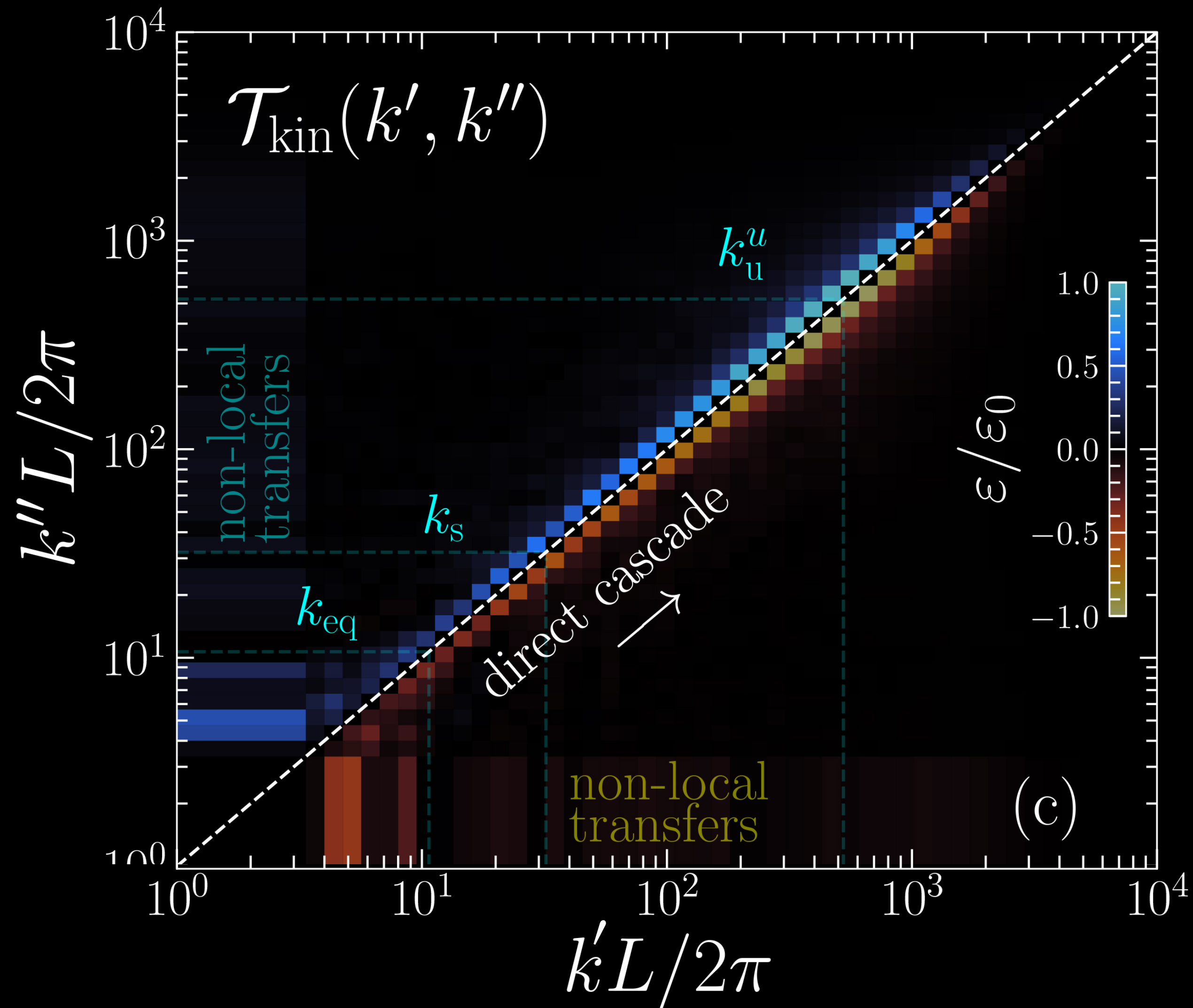
kinetic cascade terms

$$T_{uu}(k', k''' | k'') = \int dV \overbrace{\mathbf{w}''' \otimes \mathbf{u}'' : \nabla \otimes \mathbf{w}' + \frac{1}{2} \mathbf{w}' \otimes \mathbf{w}''' : (\nabla \cdot \mathbf{u}'')}^{\text{kinetic cascade terms}}$$

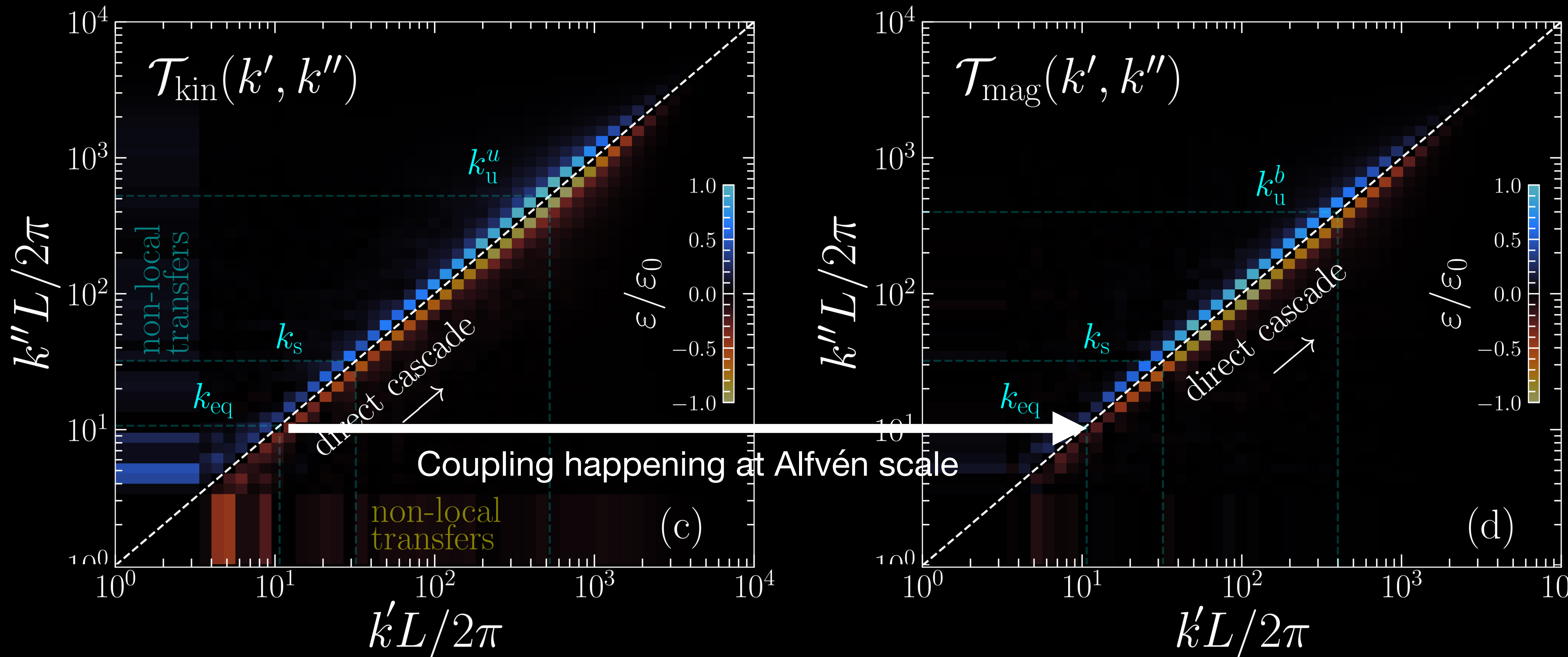
magnetic cascade terms

$$T_{bb}(k', k''' | k'') = \int dV \overbrace{\mathbf{b}''' \otimes \mathbf{u}'' : \nabla \otimes \mathbf{b}' + \frac{1}{2} \mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}'')}^{\text{magnetic cascade terms}}$$

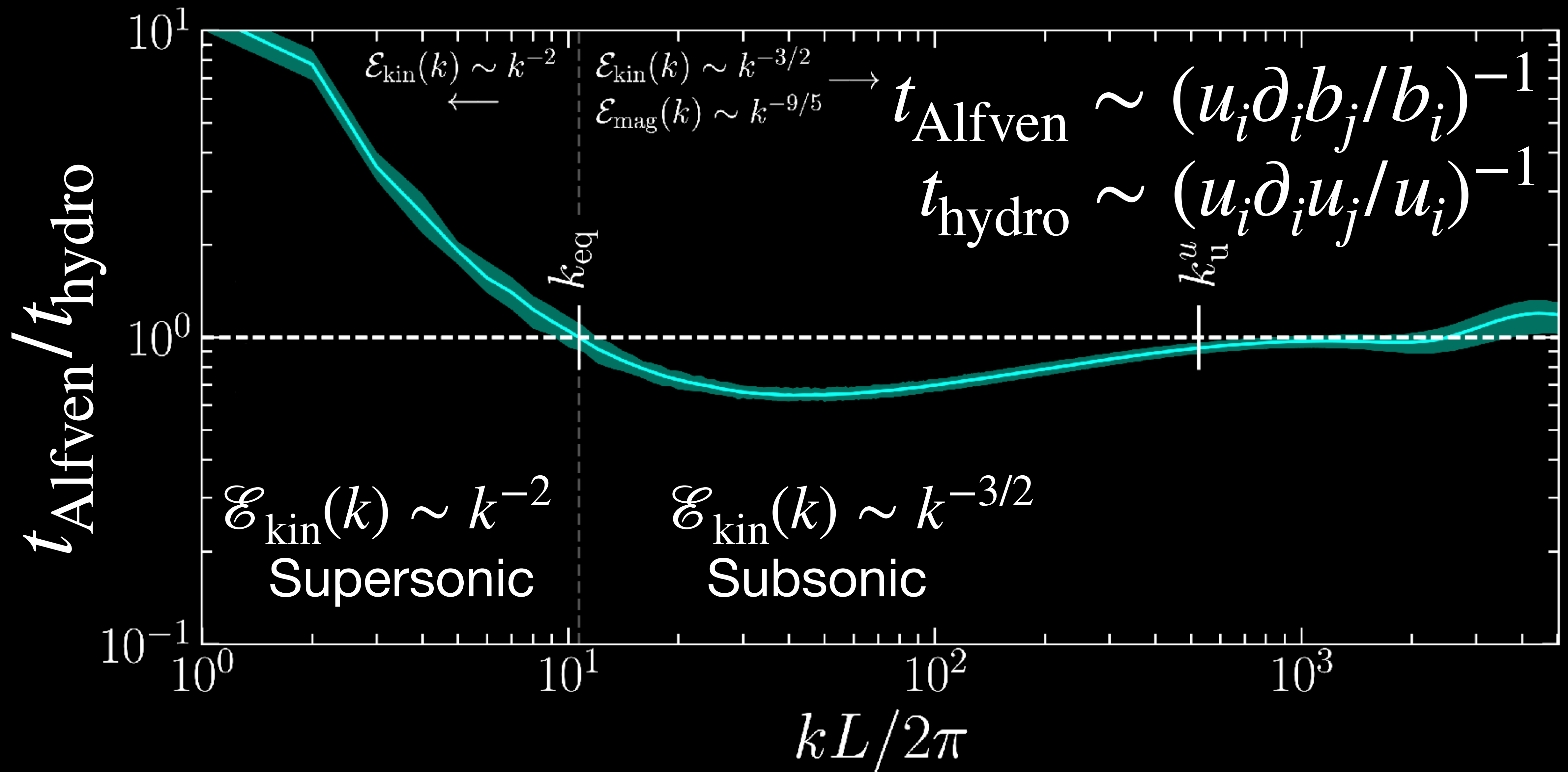
Energy spectra: the disparate lives of u and b fluctuations



Energy spectra: the disparate lives of u and b fluctuations

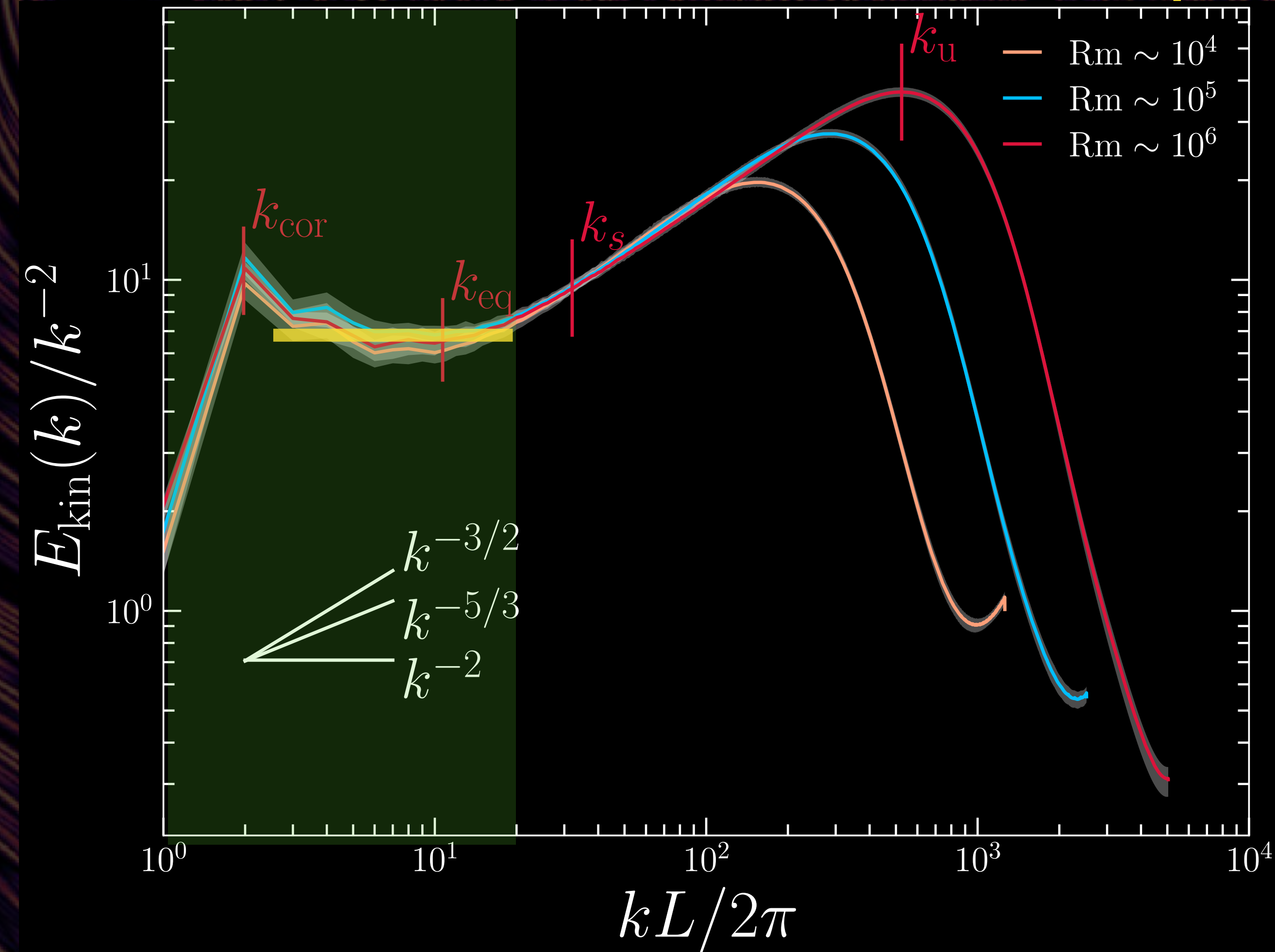


Energy spectra: the disparate lives of u and b fluctuations



The supersonic spectrum

$$\mathcal{E}(k) \sim k^{-2}, k \leq k_{\text{eq}}$$



Burgers turbulence
“Burgulence”

$$\mathcal{E}(k) \sim k^{-2}$$

**Randomly orientated
discontinuities**

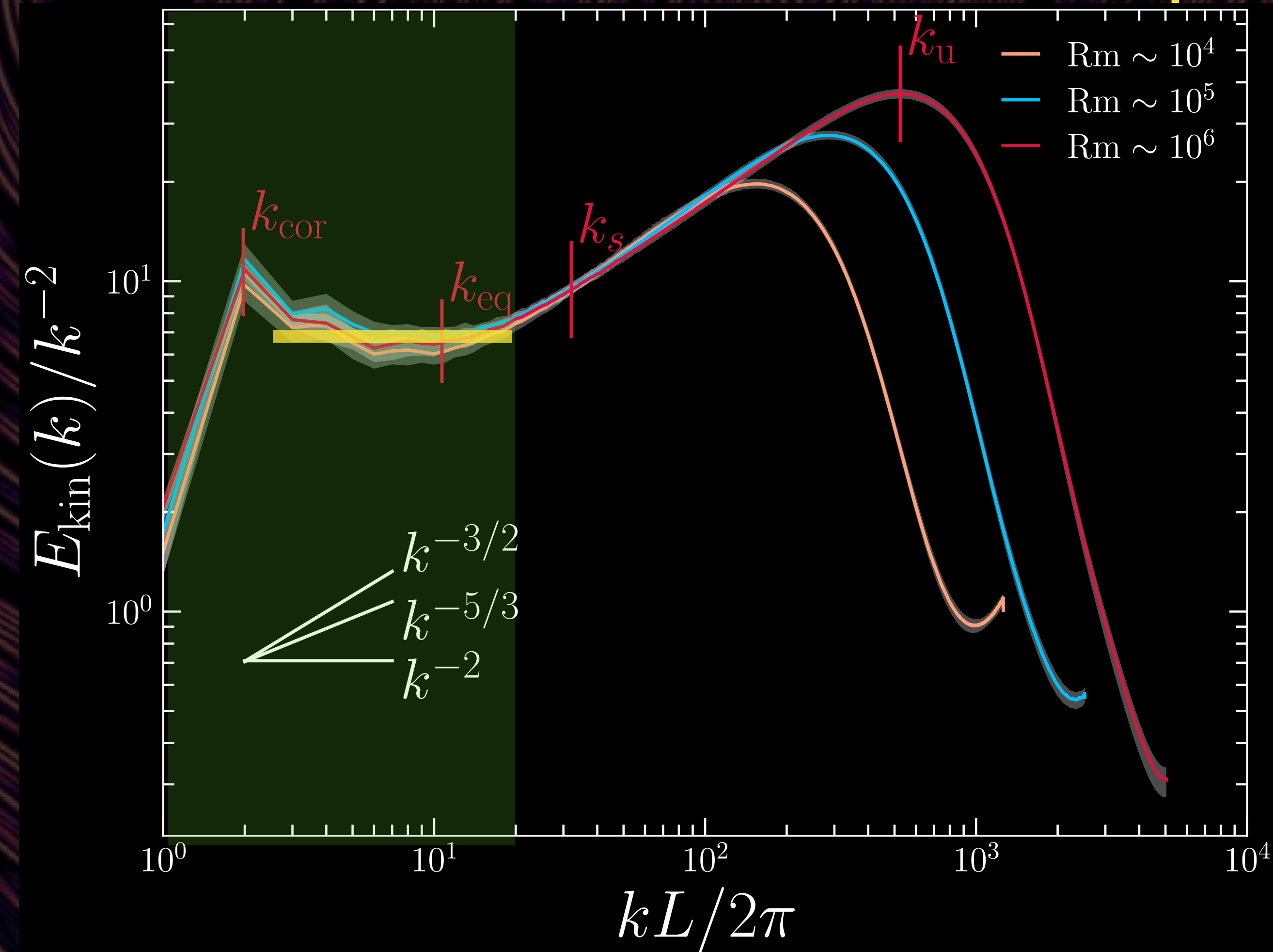
Federrath (2016)

Mocz & Burkhardt (2019)

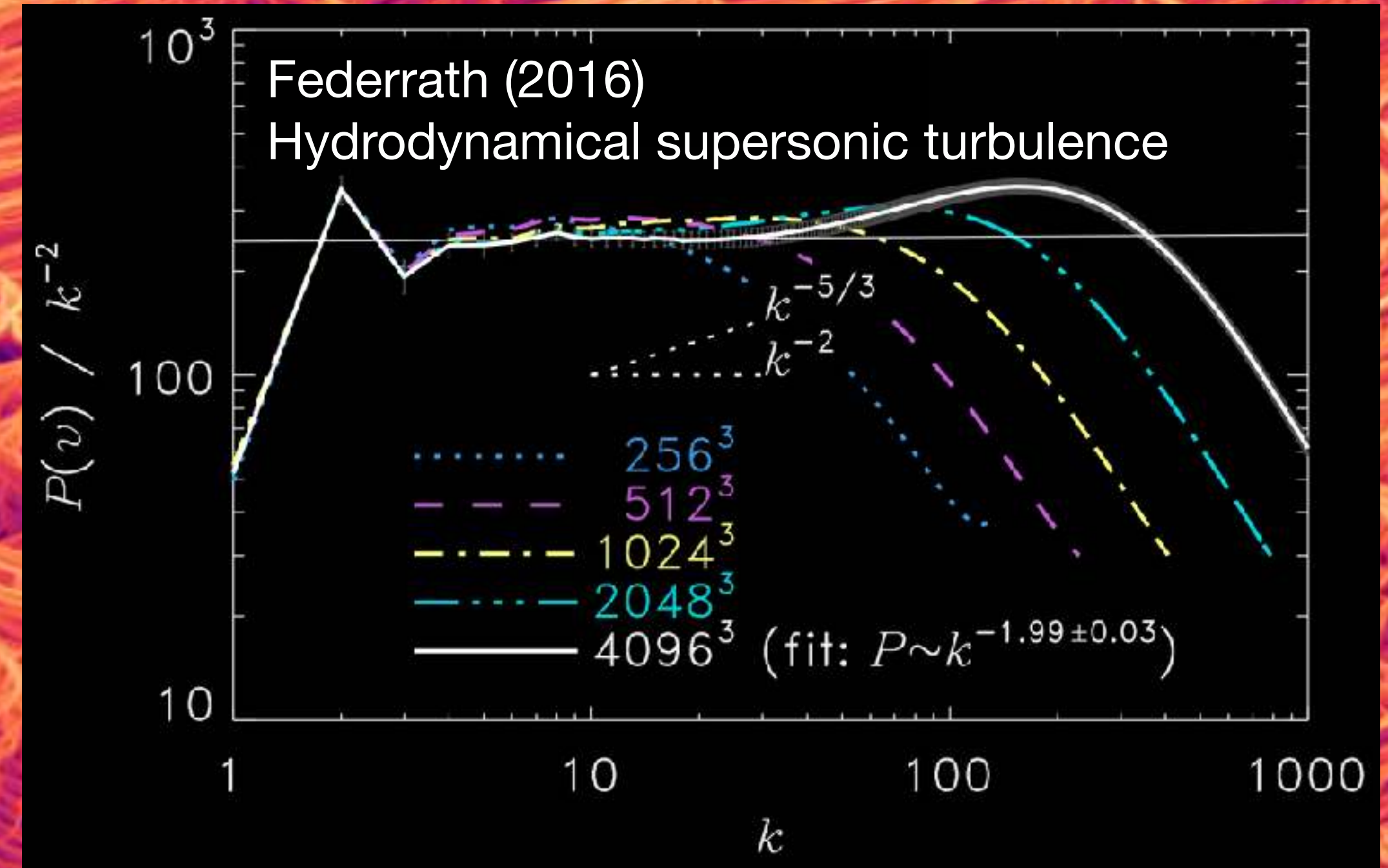
Beattie+(2022)

The supersonic spectrum

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Burgers turbulence
“Burgulence”



$$\mathcal{E}(k) \sim k^{-2}$$

**Randomly orientated
discontinuities**

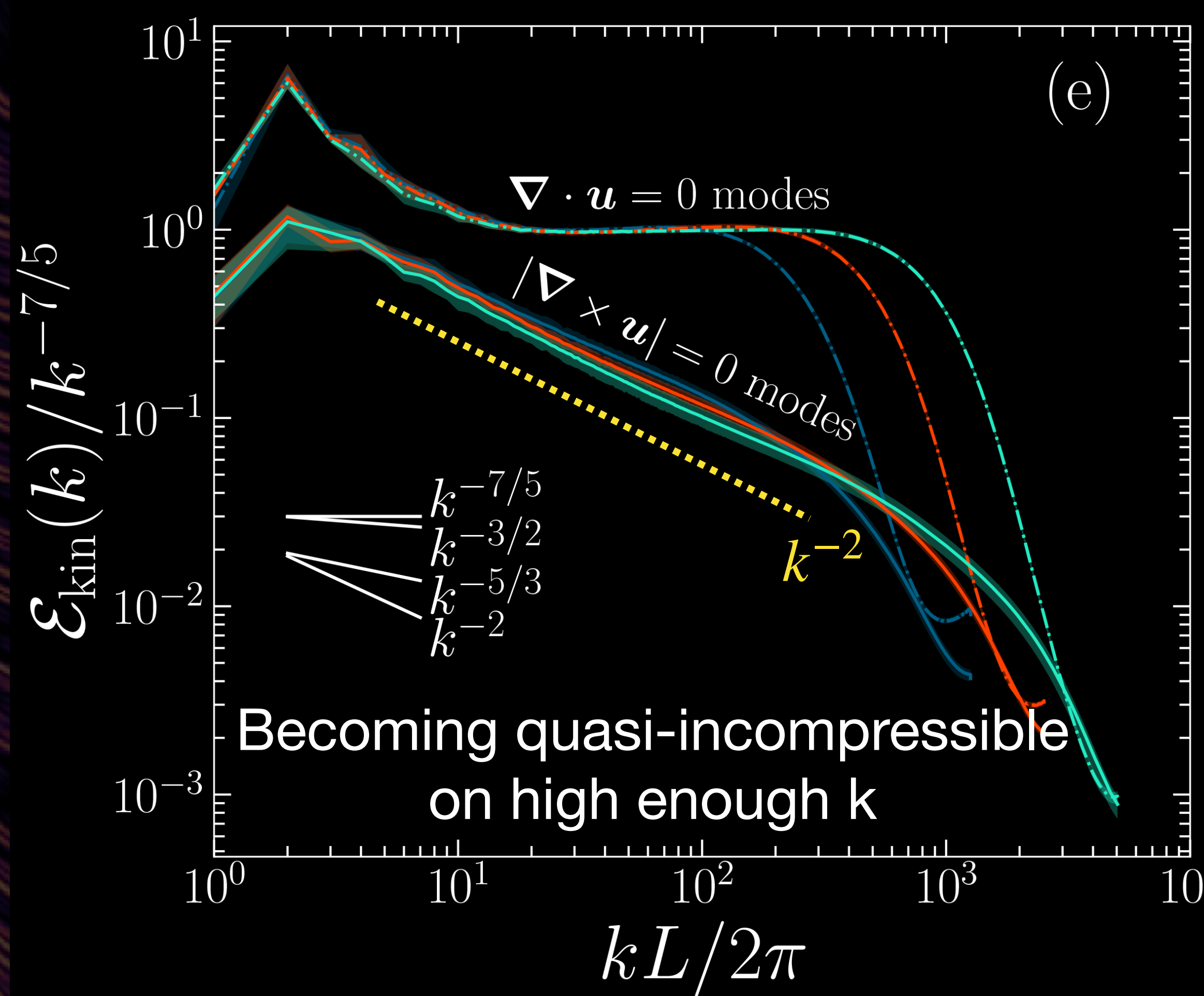
Federrath (2016)

Mocz & Burkhart (2019)

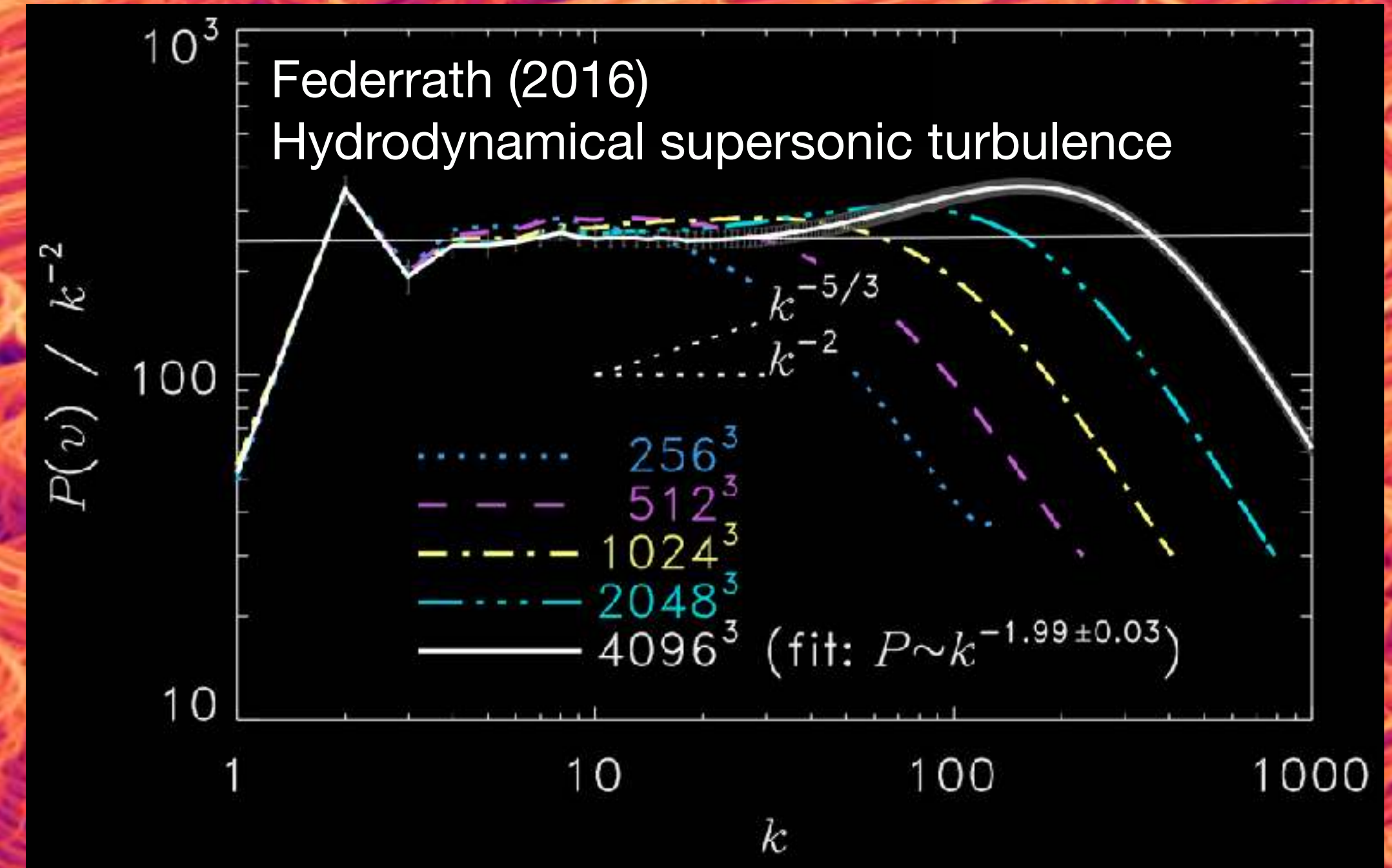
Beattie+(2022)

The supersonic spectrum

$$\mathcal{E}(k) \sim k^{-2}, k \leq k_{\text{eq}}$$



Burgers turbulence “Burgulence”



$$\mathcal{E}(k) \sim k^{-2}$$

**Randomly orientated
discontinuities**

Federrath (2016)

Mocz & Burkhart (2019)

Beattie+(2022)

The subsonic spectrum (also $\sim$ incompressible)

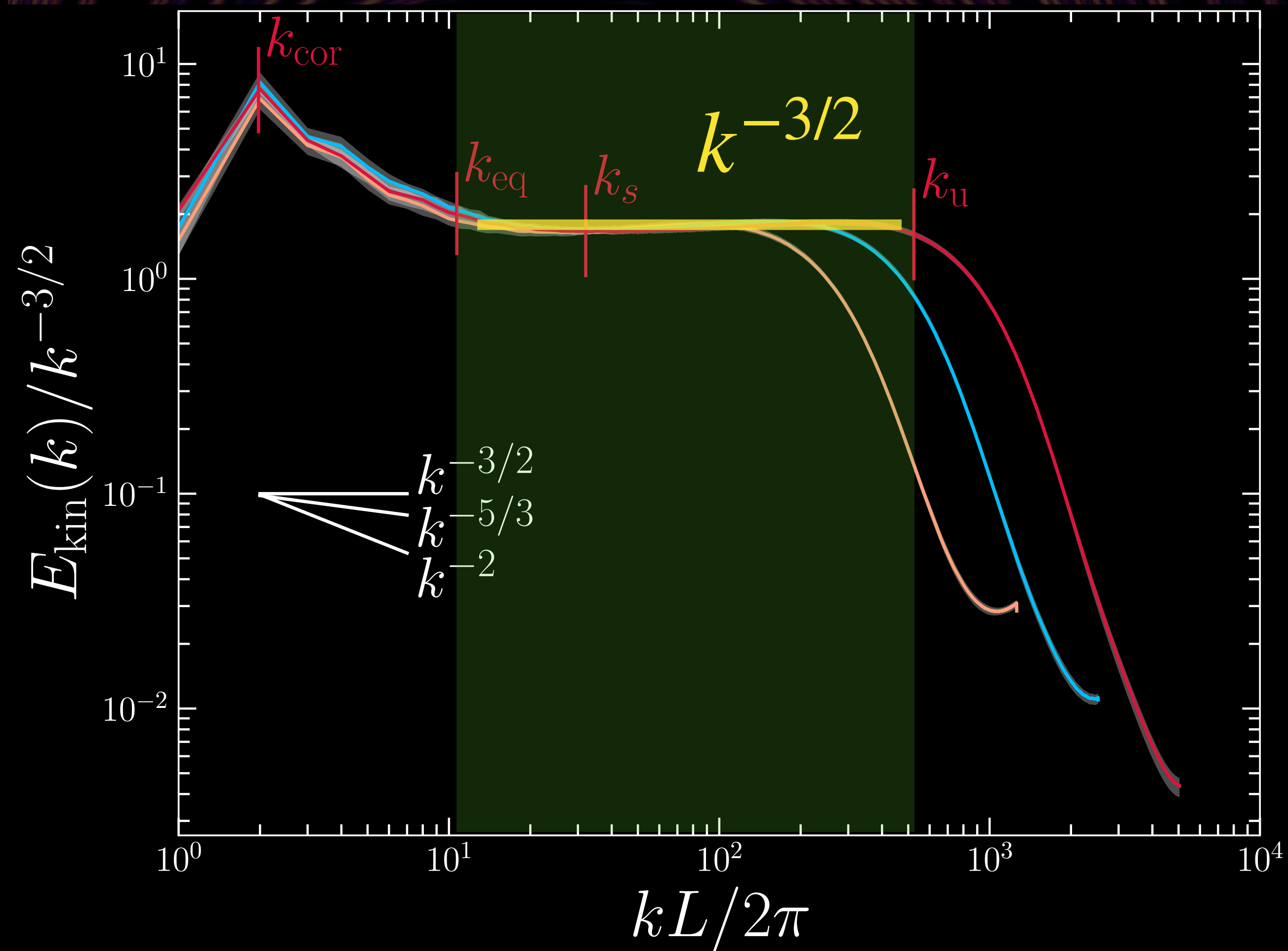
$$\mathcal{E}(k) \sim k^{-3/2}, \quad k > k_{\text{eq}}$$

Unlikely fast-wave turbulence...
Galtier (2023)

Scale-dependent alignment?
?

In general, we need:

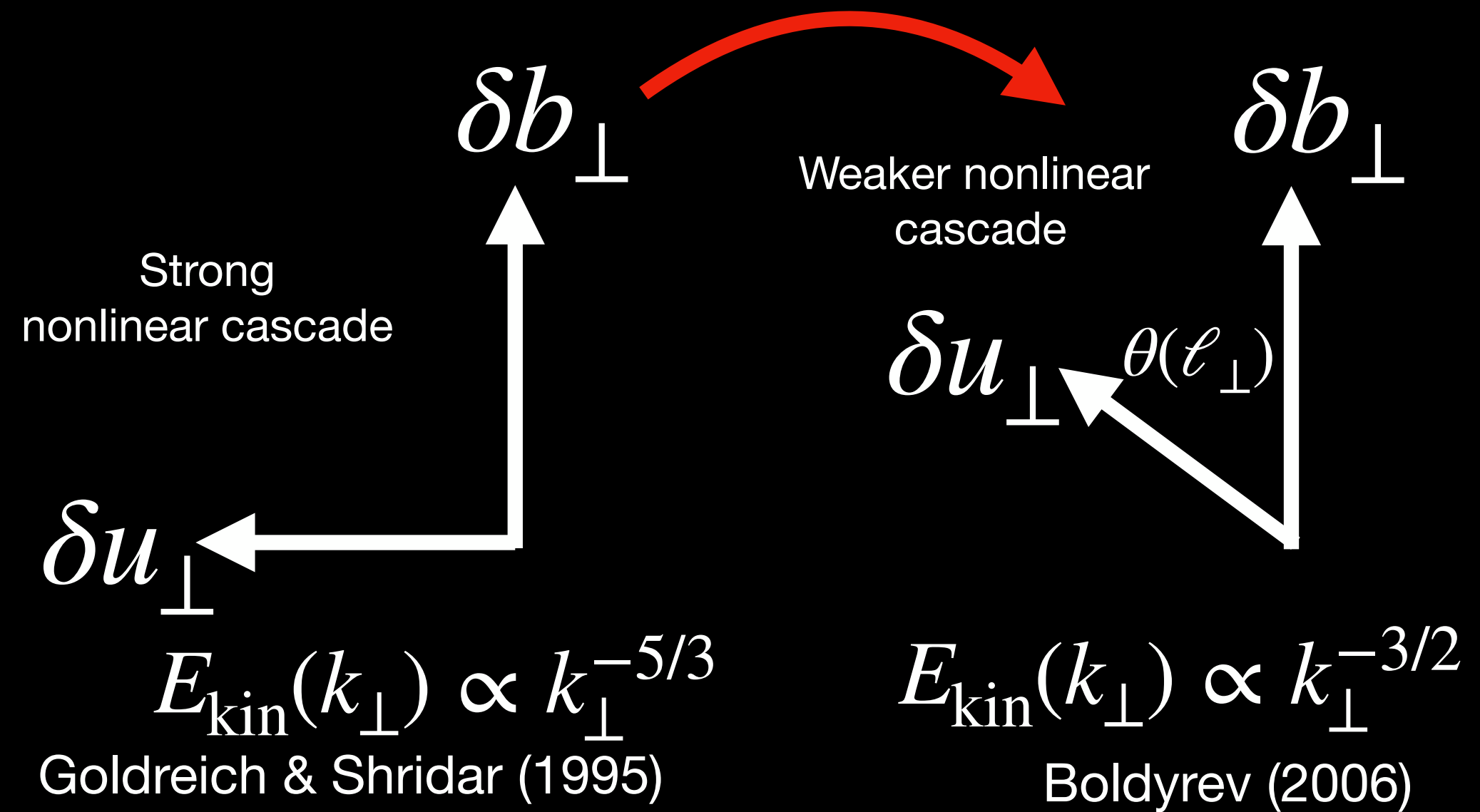
A mechanism for depleting nonlinearities
in the turbulence (turning $k^{-5/3} \rightarrow k^{-3/2}$)





Dynamic alignment?

Shearing events between counter-propagating Alfvén waves / selective decay

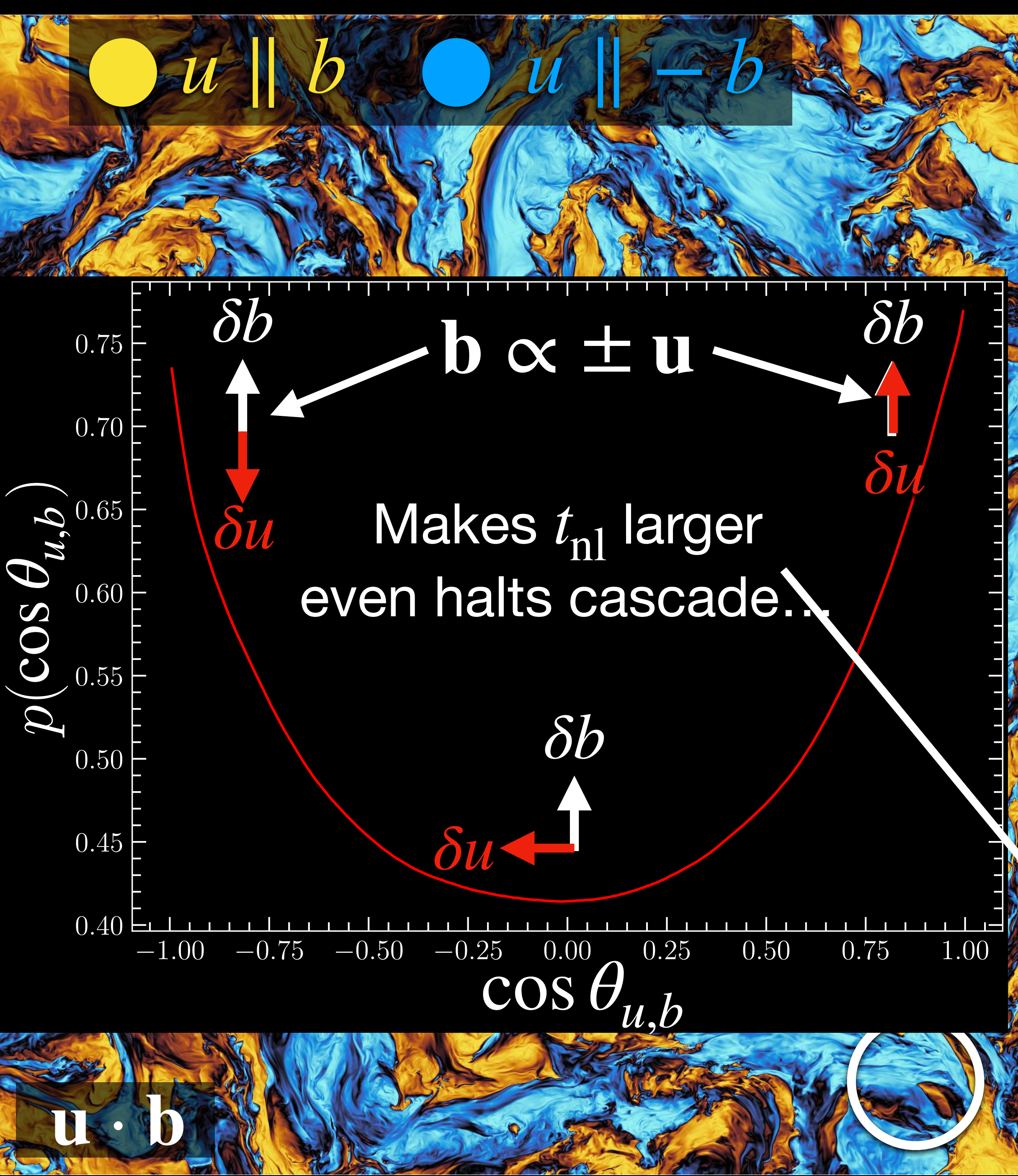


Modifies the cascade timescale

$$t_{\text{nl}} \sim \frac{\ell_{\perp}}{z^{\mp} \sin \theta_{z^{\mp}}}, \quad z^{\mp} = (u \mp b)$$

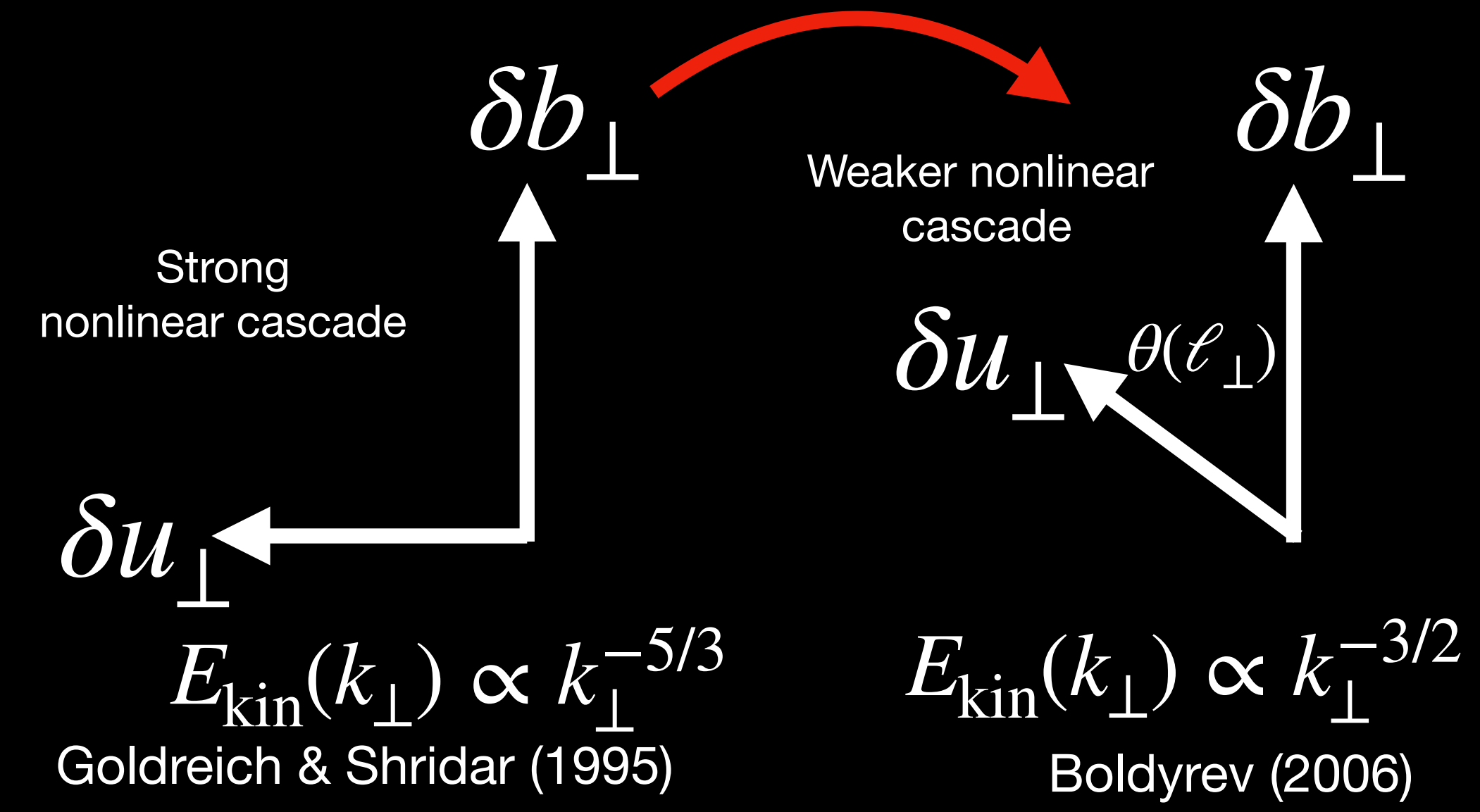
Boldyrev (2006)

$\theta(\ell_{\perp}) \sim \ell_{\perp}^{1/4}$



Dynamic alignment?

Shearing events between counter-propagating Alfvén waves / selective decay



Modifies the cascade timescale

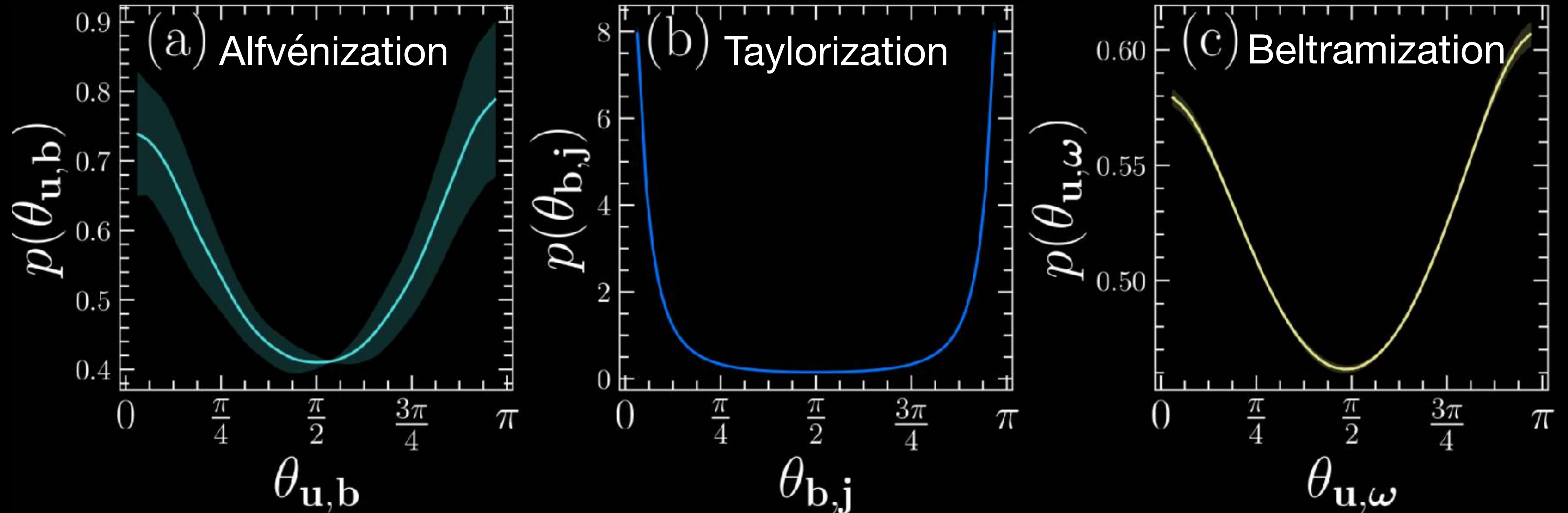
$$t_{\text{nl}} \sim \frac{\ell_{\perp}}{z^{\mp} \sin \theta_{z^{\mp}}}, \quad z^{\mp} = (u \mp b)$$

Boldyrev (2006)

$\theta(\ell_{\perp}) \sim \ell_{\perp}^{1/4}$

But even more alignment than just $\mathbf{u}$ and $\mathbf{b}$
 Searching to weaken the nonlinearities

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}$$

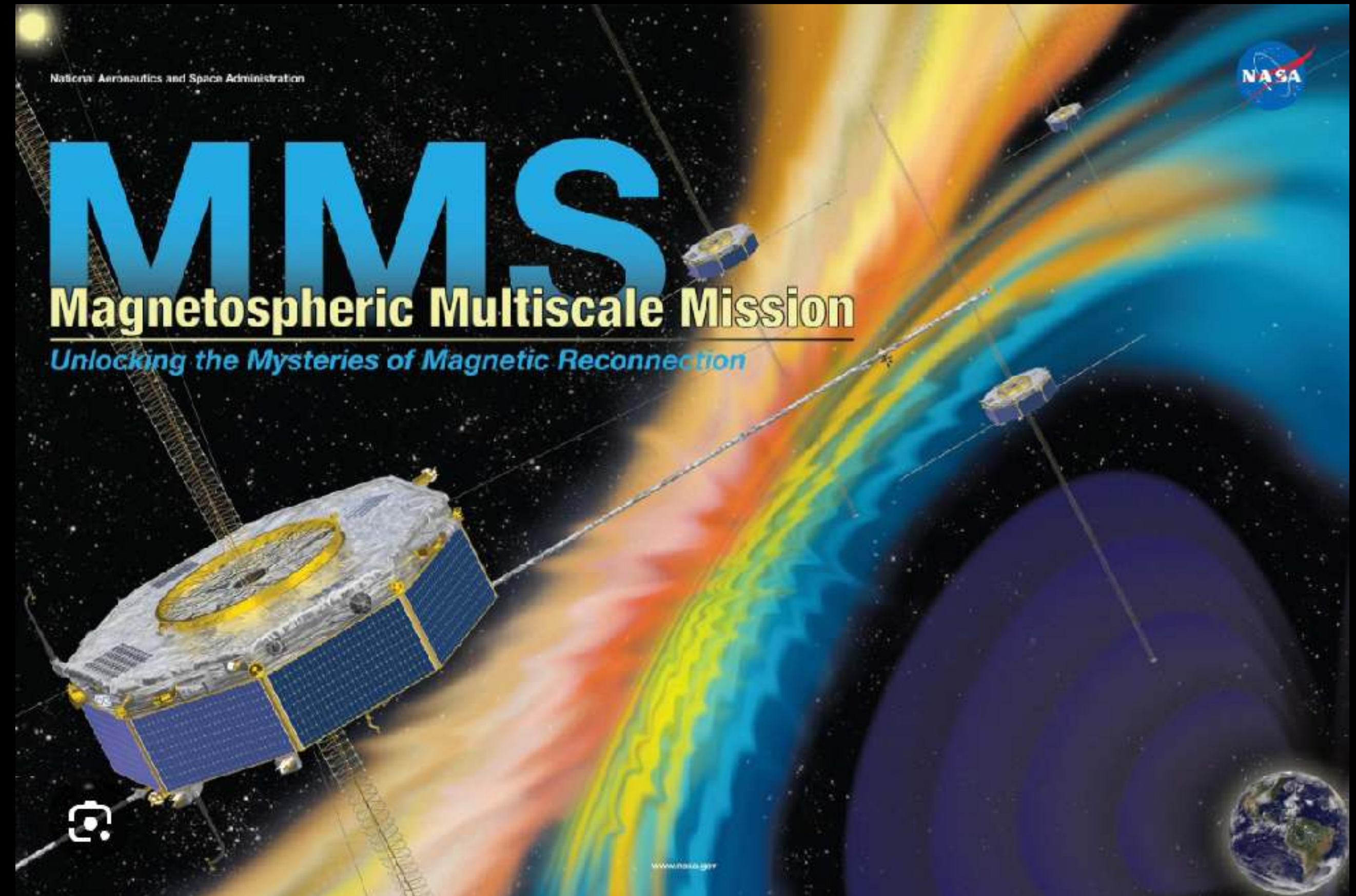
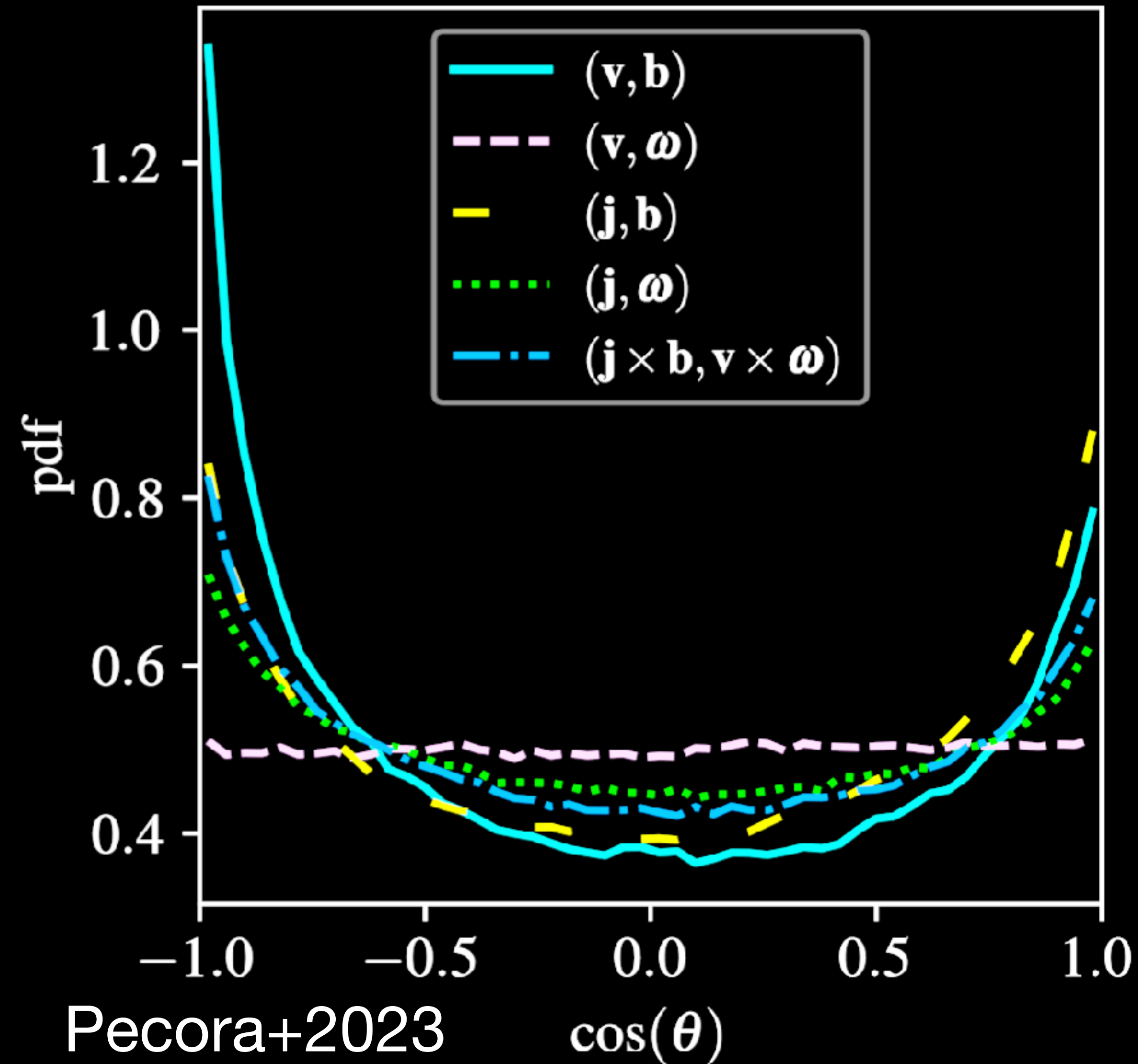


looks like a relaxation problem

$$\mathbf{u} \propto \mathbf{b} \propto \mathbf{j} \propto \boldsymbol{\omega}$$

$$\delta_k \left(u^2(k) + b^2(k) - \lambda_{k,1} b(k) \cdot a^\dagger(k) - \lambda_{k,2} u(k) \cdot b^\dagger(k) \right) = 0$$

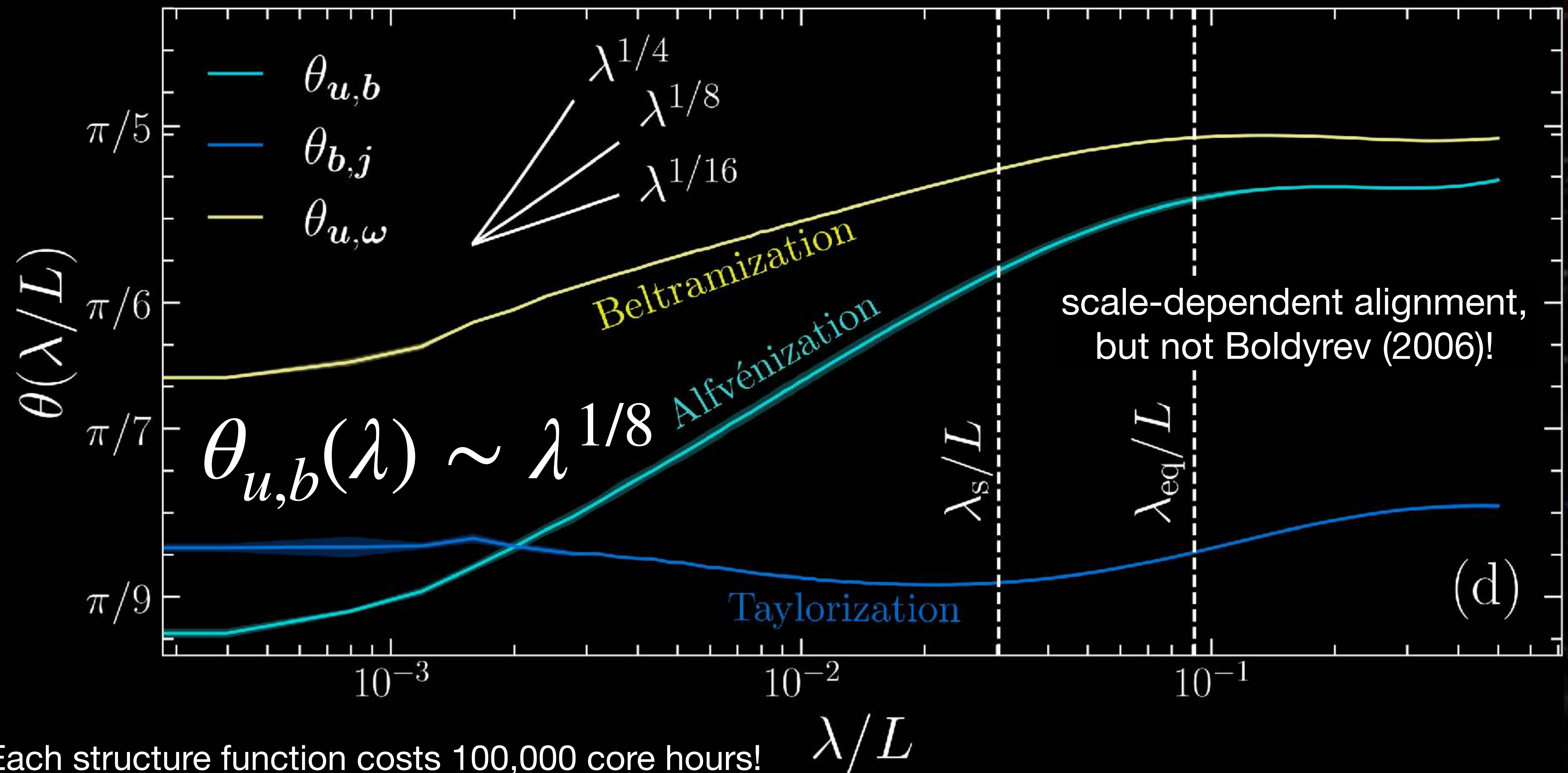
But even more alignment than just $\mathbf{u}$ and $\mathbf{b}$
Searching to weaken the nonlinearities



Highly-aligned states
in magnetosheath
turbulence!

Alignment not happening on all scales...

Beattie & Bhattacharjee (subm. PRL.).
Scale dependent alignment in compressible
magnetohydrodynamic turbulence



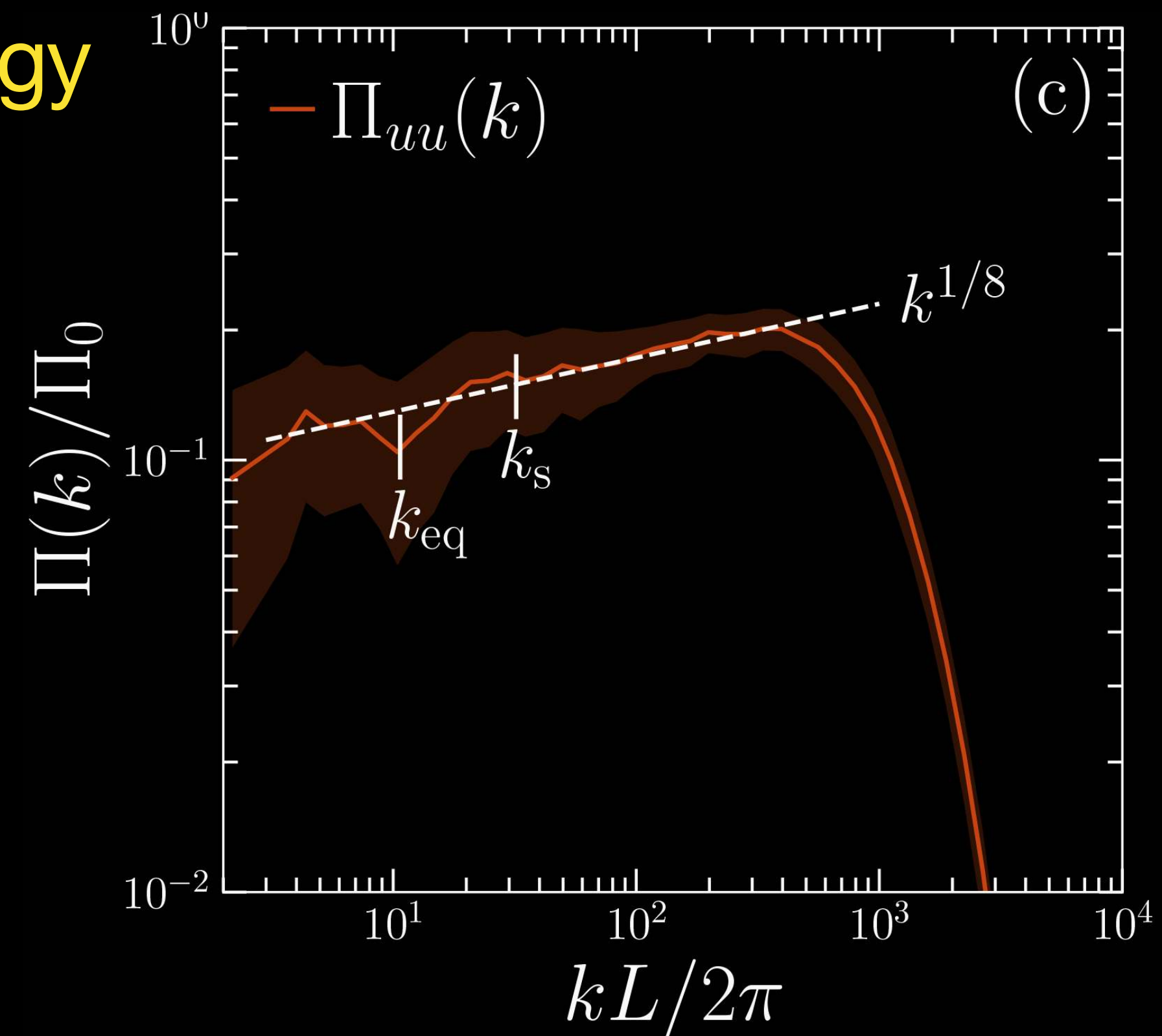
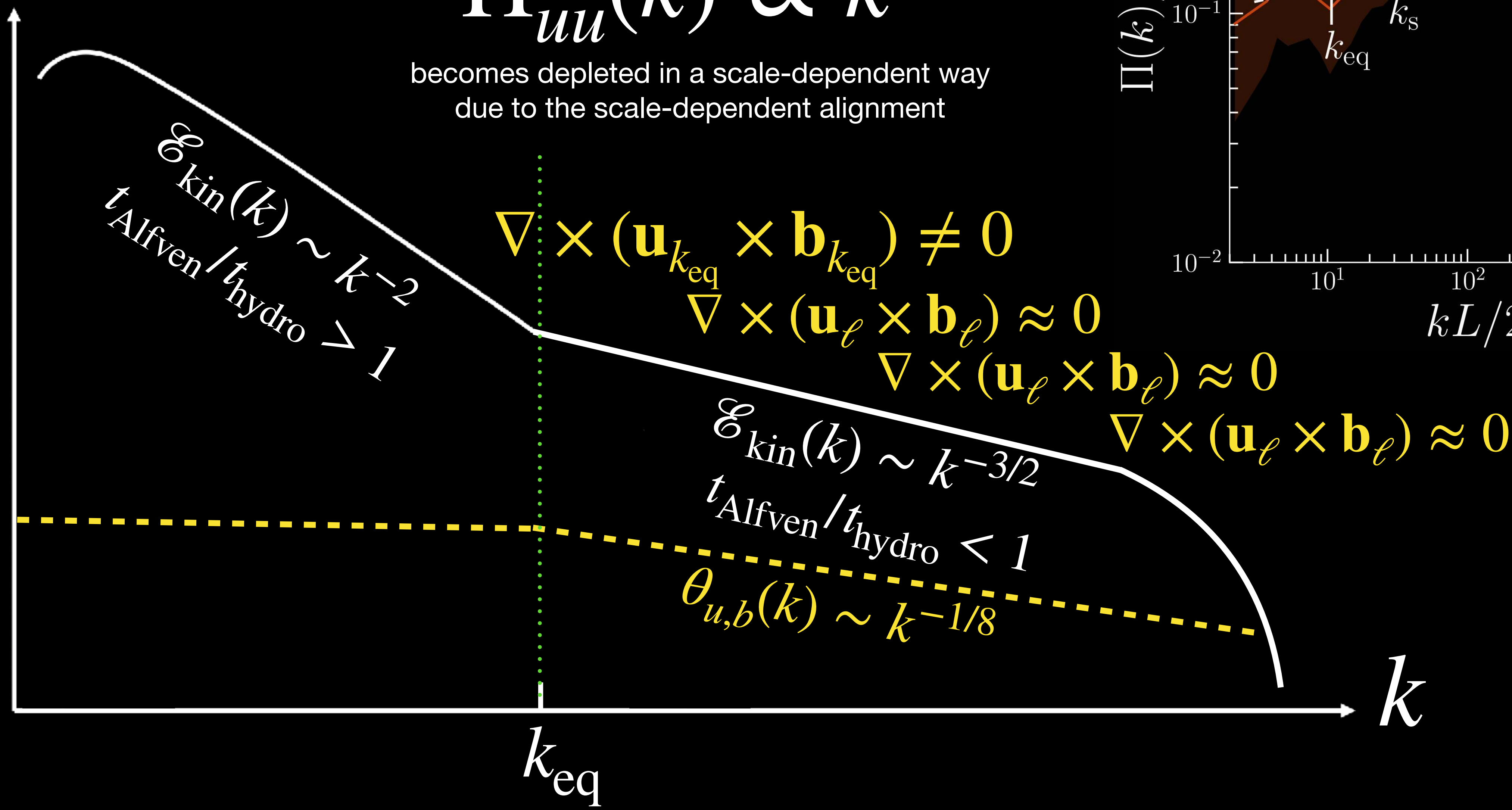
Each structure function costs 100,000 core hours!

The subsonic spectrum — a whole new phenomenology

Depletion through dynamo fluxes (not Boldeyrev 2006)

$$\Pi_{uu}(k) \propto k^{1/8}$$

becomes depleted in a scale-dependent way
due to the scale-dependent alignment



Putting it all together to understand the kinetic spectrum

Adding alignment to spectrum as in Boldeyrev (2006)

$$E_{\text{kin}}(k) \propto \Pi_{uu}^{2/3} \theta_{u,b}^{-2/3} k^{-5/3},$$

modified nonlinear timescale

with

$$\theta_{u,b}(\ell) \sim \ell^{1/8}, \quad \Pi_{uu}(k) \propto k^{1/8}$$

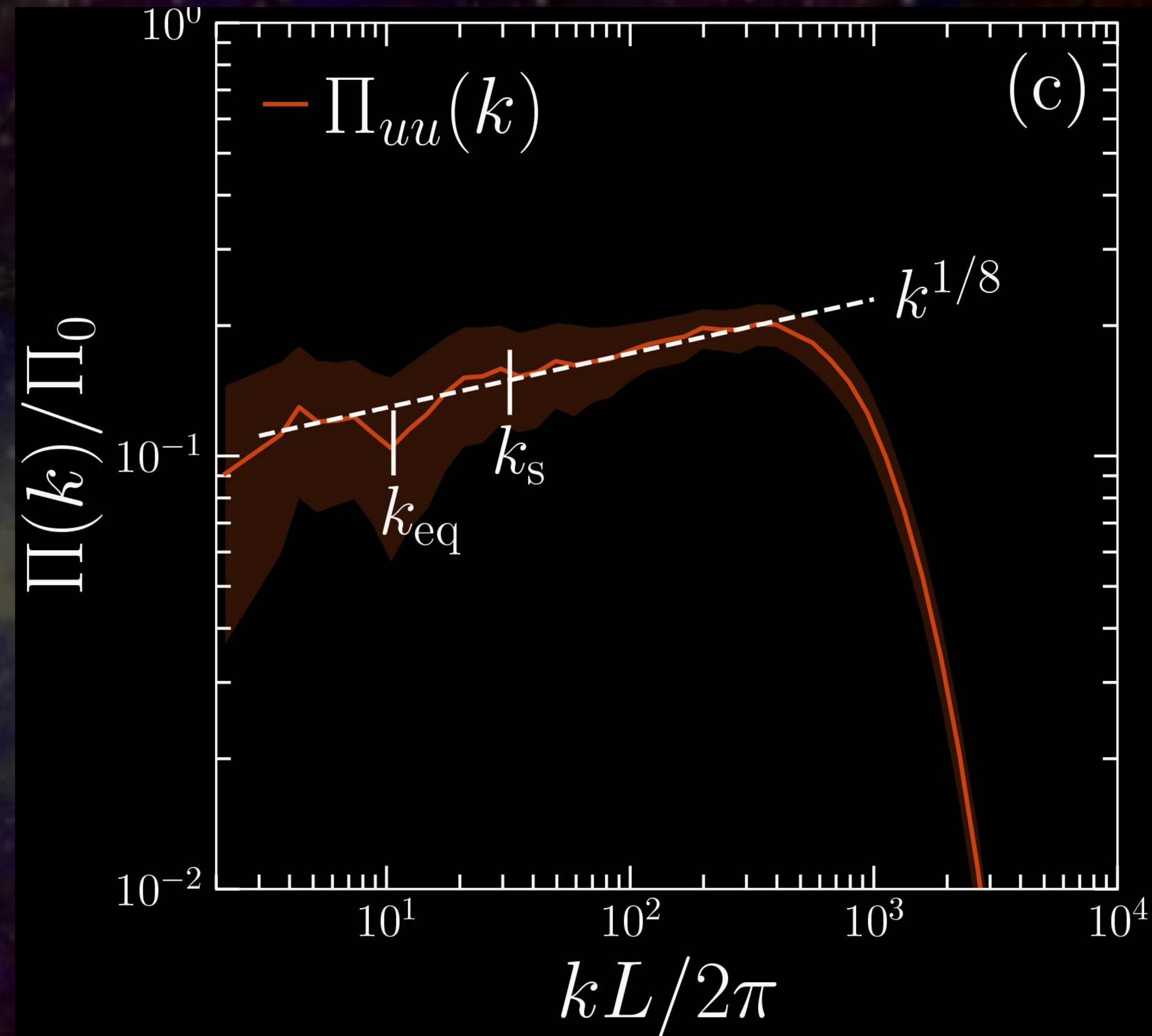
u, b coupling

depletion of UU reservoir

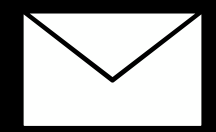
implies

$$E_{\text{kin}}(k) \propto k^{-3/2},$$

Constancy of energy flux within a single spectrum is violated, making the cascade less nonlinear!



Thanks, questions?



james.beattie@princeton.edu



[@astro_magnetism](https://twitter.com/astro_magnetism)

The future is bright

- Next generation of $10,000^3 +$ simulations with large-scale field integrating now
- Moving to also have AthenaK capabilities to do GPU counterpart runs
- Pushing towards $20,000^3$ (500TB checkpoints) to further explore asymptotic state

Extra slides

Putting it all together to understand the kinetic spectrum

Consider

$$E_{\text{kin}} \propto \Pi_{uu}^{2/3} \theta_{u,b}^{-2/3} k^{-5/3}, \quad \theta_{u,b}(\ell) \sim \ell^{1/8}$$

which means

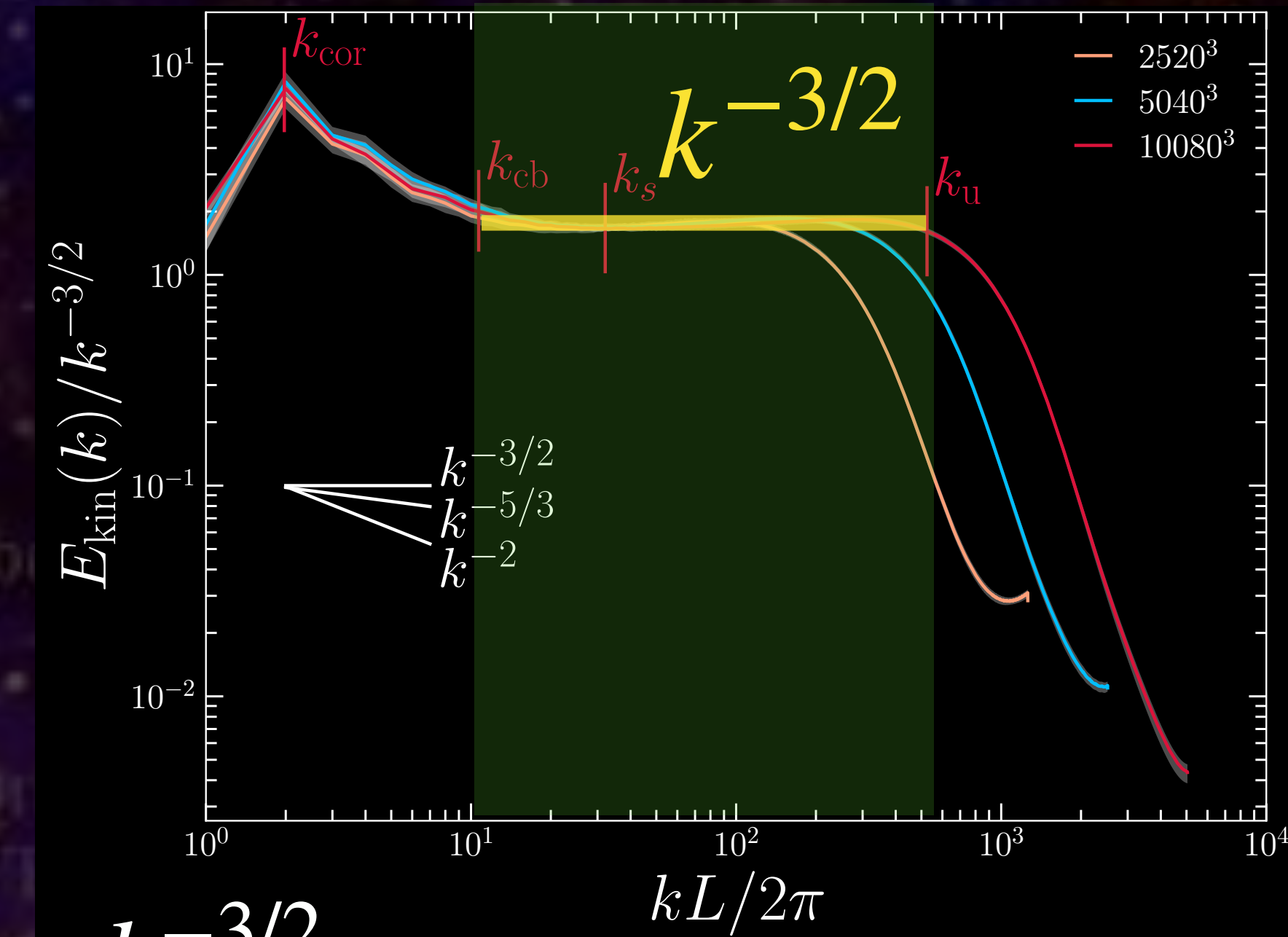
$$E_{\text{kin}} \propto \Pi_{uu}^{2/3} k^{-19/12} \propto k^{1.58},$$

inconsistent... not strong enough to suppress $k^{-5/3} \rightarrow k^{-3/2}$

Crazy idea... what if....

$$\Pi_{uu}(\ell) \propto \ell^\beta$$

Very anti-Kolmogorov. Let's measure.



We find

$$\theta(\ell_{\perp}) \sim \ell_{\perp}^{1/8}$$

scale-dependent alignment,
but not Boldyrev (2006).

The turbulent plasma wants to rid itself of all nonlinearities

Compressible MHD equations

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\partial_t(\rho \mathbf{u}) + \nabla \cdot \left(\cancel{\rho \mathbf{u} \otimes \mathbf{u}} + p \mathbb{I} - \frac{1}{4\pi} \cancel{\mathbf{b} \otimes \mathbf{b}} \right) = \rho \mathbf{f} + \nabla \cdot \mathbb{D}_\nu(\rho \mathbf{u})$$

$\mathbf{u} \times \boldsymbol{\omega}$ $\mathbf{j} \times \mathbf{b}$

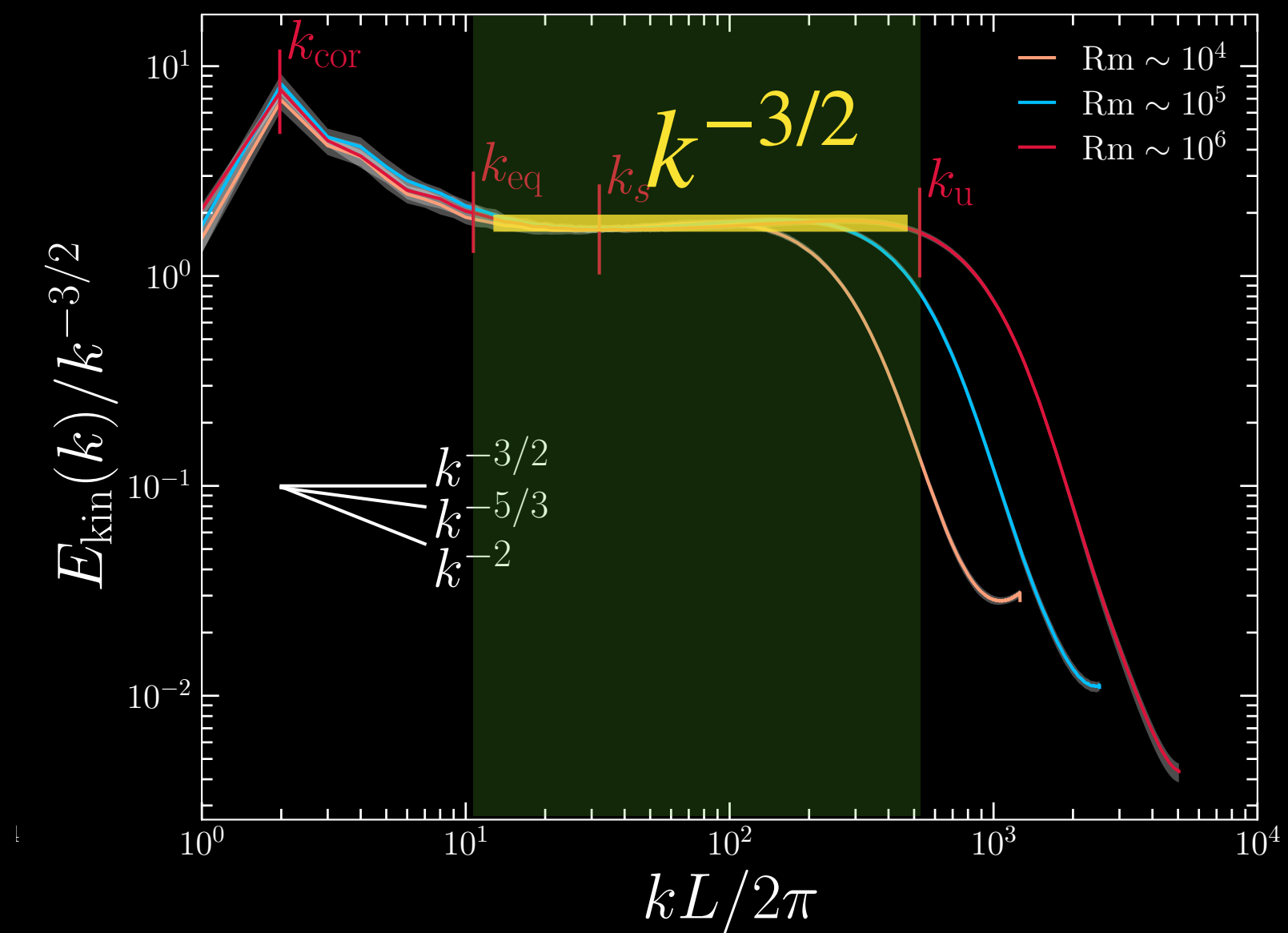
$$\partial_t \mathbf{b} + \nabla \cdot (\cancel{\mathbf{u} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{u}}) = \nabla \cdot \mathbb{D}_\eta(\mathbf{b})$$

$\mathbf{u} \times \mathbf{b}$

$$\nabla \cdot \mathbf{b} = 0$$

$$p = c_s^2 \rho + \frac{1}{8\pi} \cancel{\mathbf{b} \cdot \mathbf{b}}$$

$\mathbf{j} \times \mathbf{b}$



Beltrami Euler equations

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

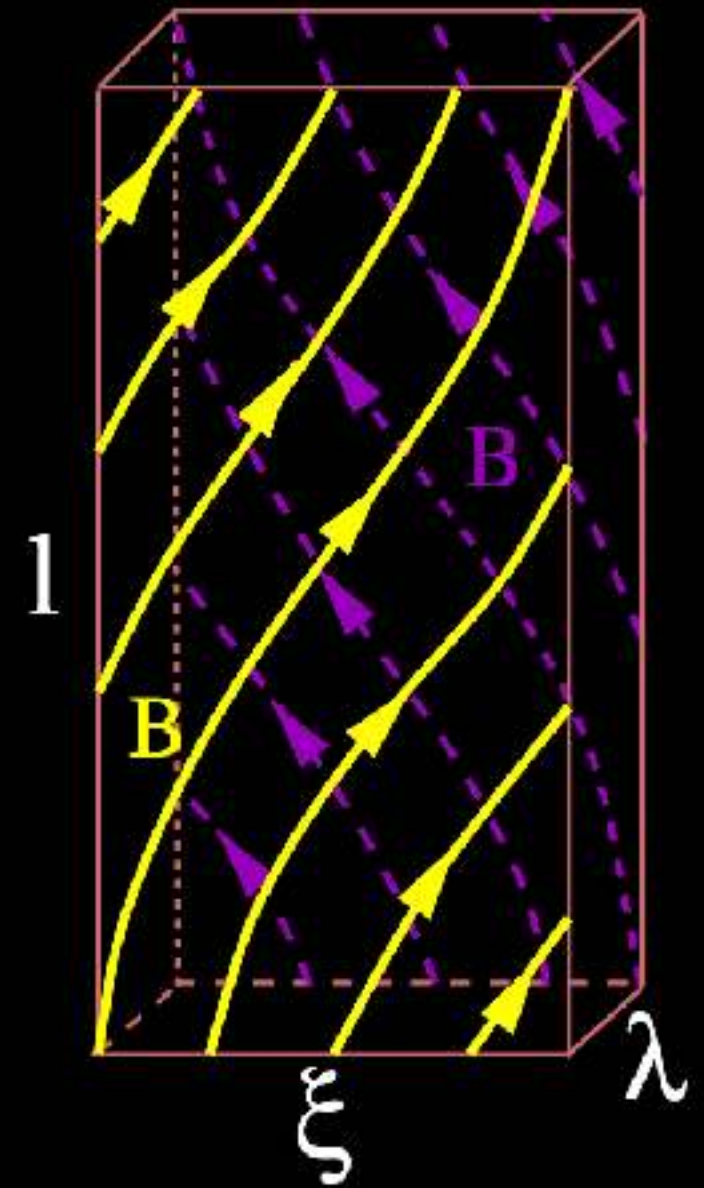
$$\partial_t(\rho \mathbf{u}) + c_s^2 \rho + \frac{1}{2} \nabla u^2 = 0$$

Dynamic alignment? Searching to weaken the nonlinearities

$$t_A \sim t_{nl}$$

Assume anisotropy is 3D, critical balance, maximal alignment preserving 3D anisotropy

Boldyrev (2006) $\theta(\ell) \sim \ell^{1/4}$



We can test this using first order structure function

$$|\theta(\ell/L)| \sim \frac{\langle |\delta \mathbf{b}_\perp \times \delta \mathbf{v}_\perp| \rangle_\ell}{\langle |\delta \mathbf{b}_\perp| |\delta \mathbf{v}_\perp| \rangle_\ell}$$

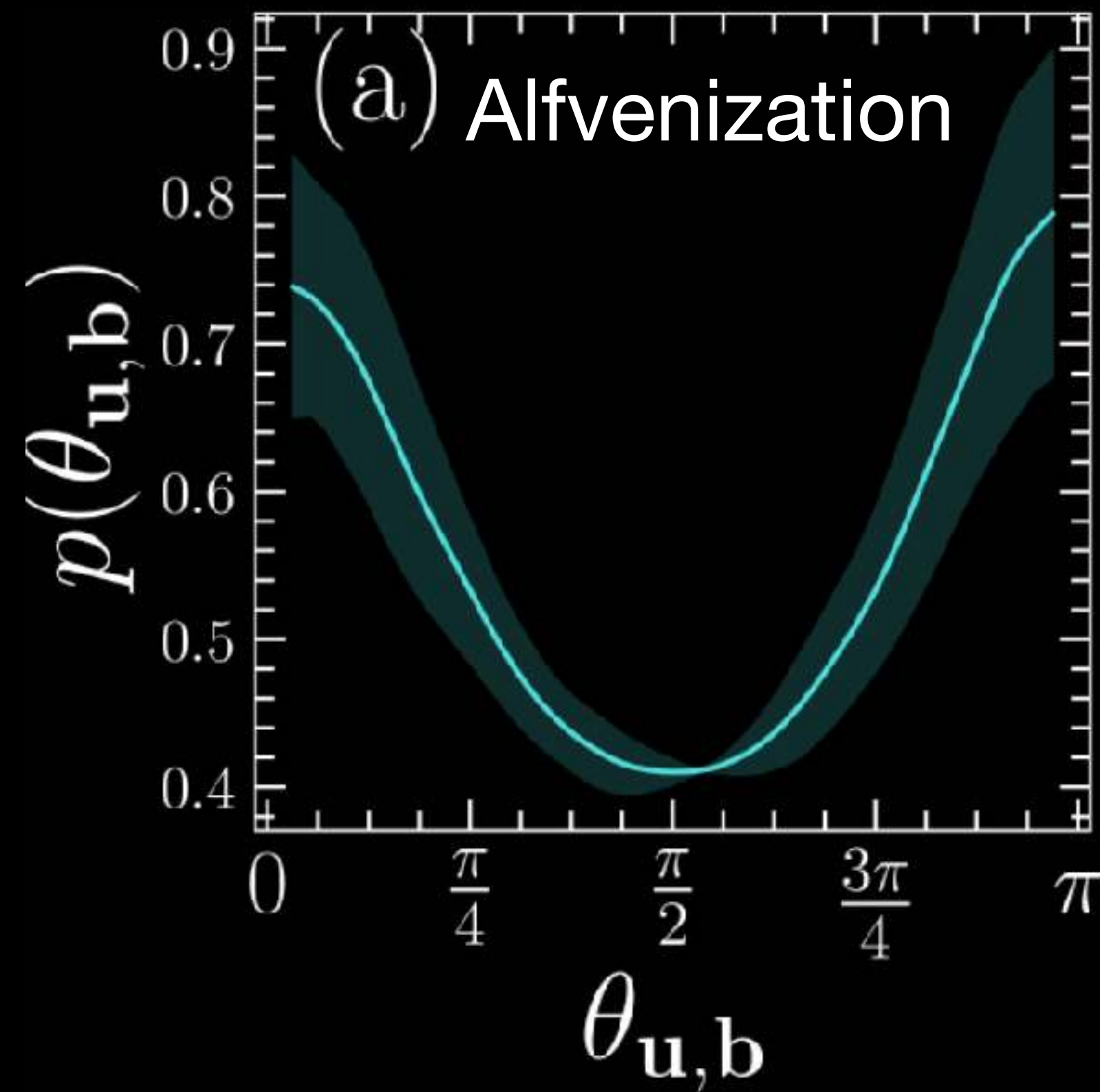
$$\delta \mathbf{b} = \mathbf{b}(\mathbf{r}) - \mathbf{b}(\mathbf{r} + \ell)$$

$$\hat{\mathbf{b}}_\ell = \frac{\mathbf{b}(\mathbf{r}) + \mathbf{b}(\mathbf{r} + \ell)}{\|\mathbf{b}(\mathbf{r}) + \mathbf{b}(\mathbf{r} + \ell)\|}$$

$$\delta \mathbf{b}_\perp = \delta \mathbf{b} - (\delta \mathbf{b} \cdot \hat{\mathbf{b}}_\ell) \hat{\mathbf{b}}_\ell$$

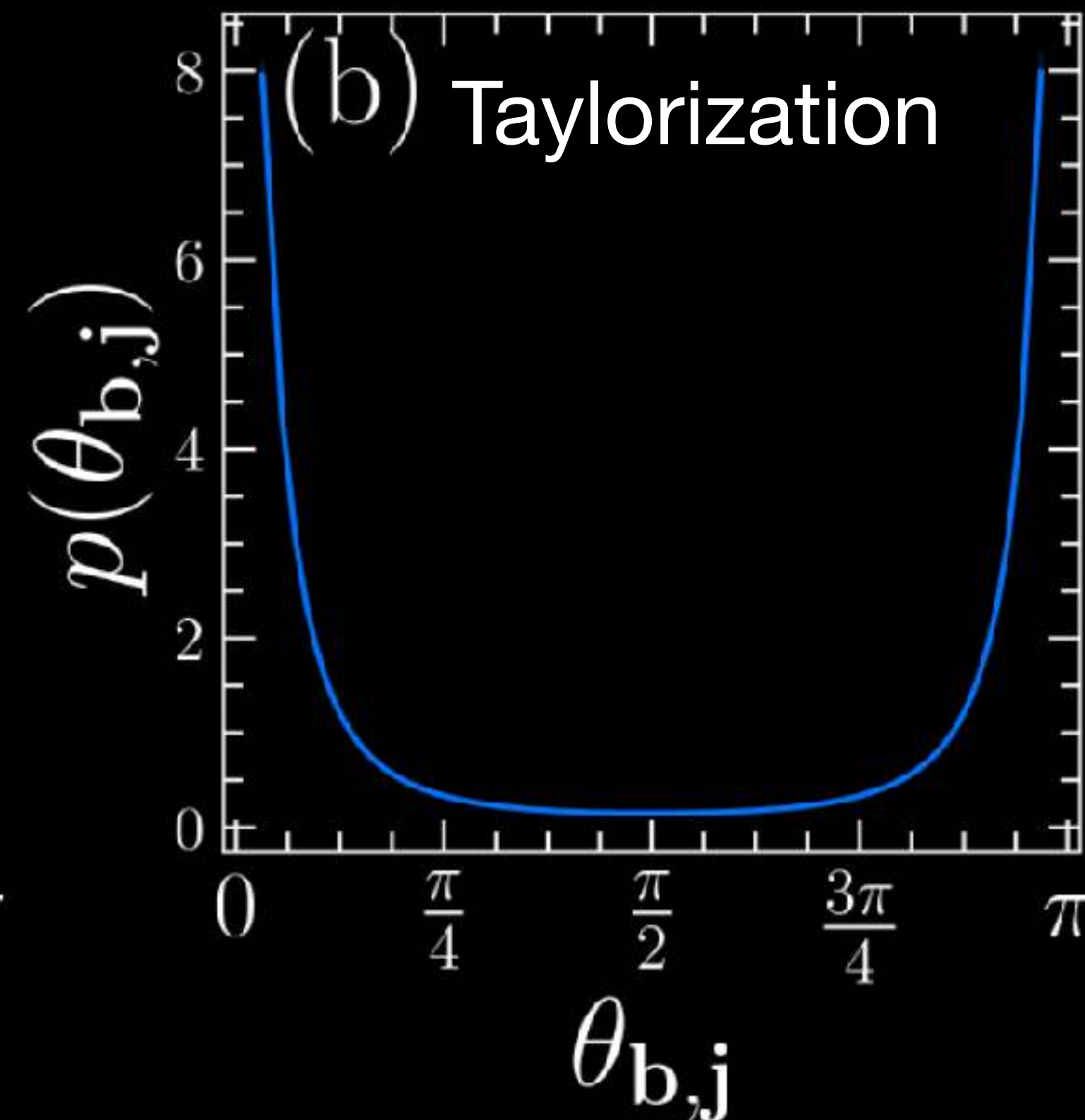
see e.g., Dong+(2018); Chernoglazov+(2021) for methods

But even more alignment than just $\mathbf{u}$ and $\mathbf{b}$
 Searching to weaken the nonlinearities



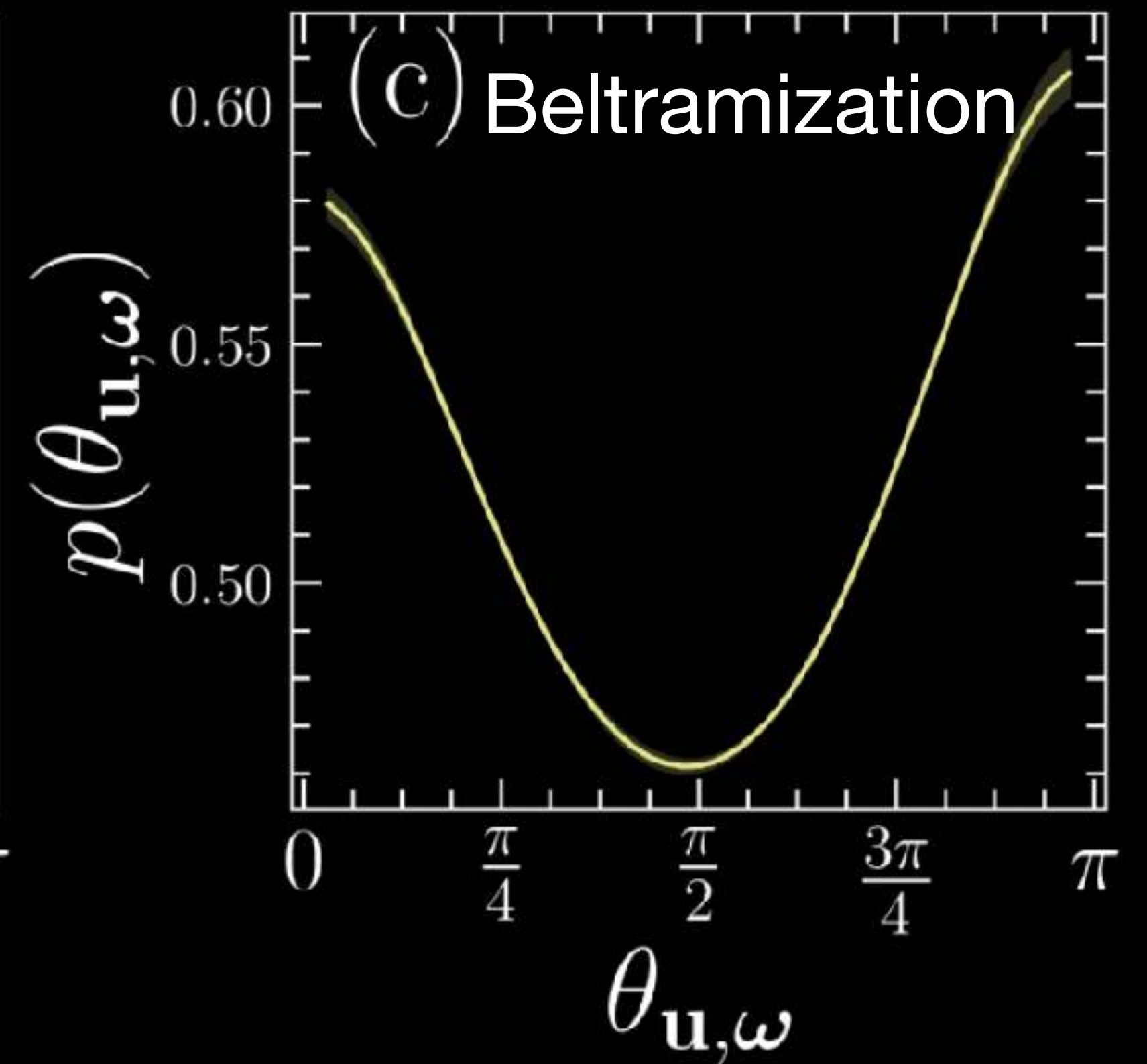
$$\nabla \times (\mathbf{u} \times \mathbf{b})$$

Induction



$$\mathbf{j} \times \mathbf{b}$$

Lorentz force



$$\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \sim \boldsymbol{\omega} \times \mathbf{u}$$

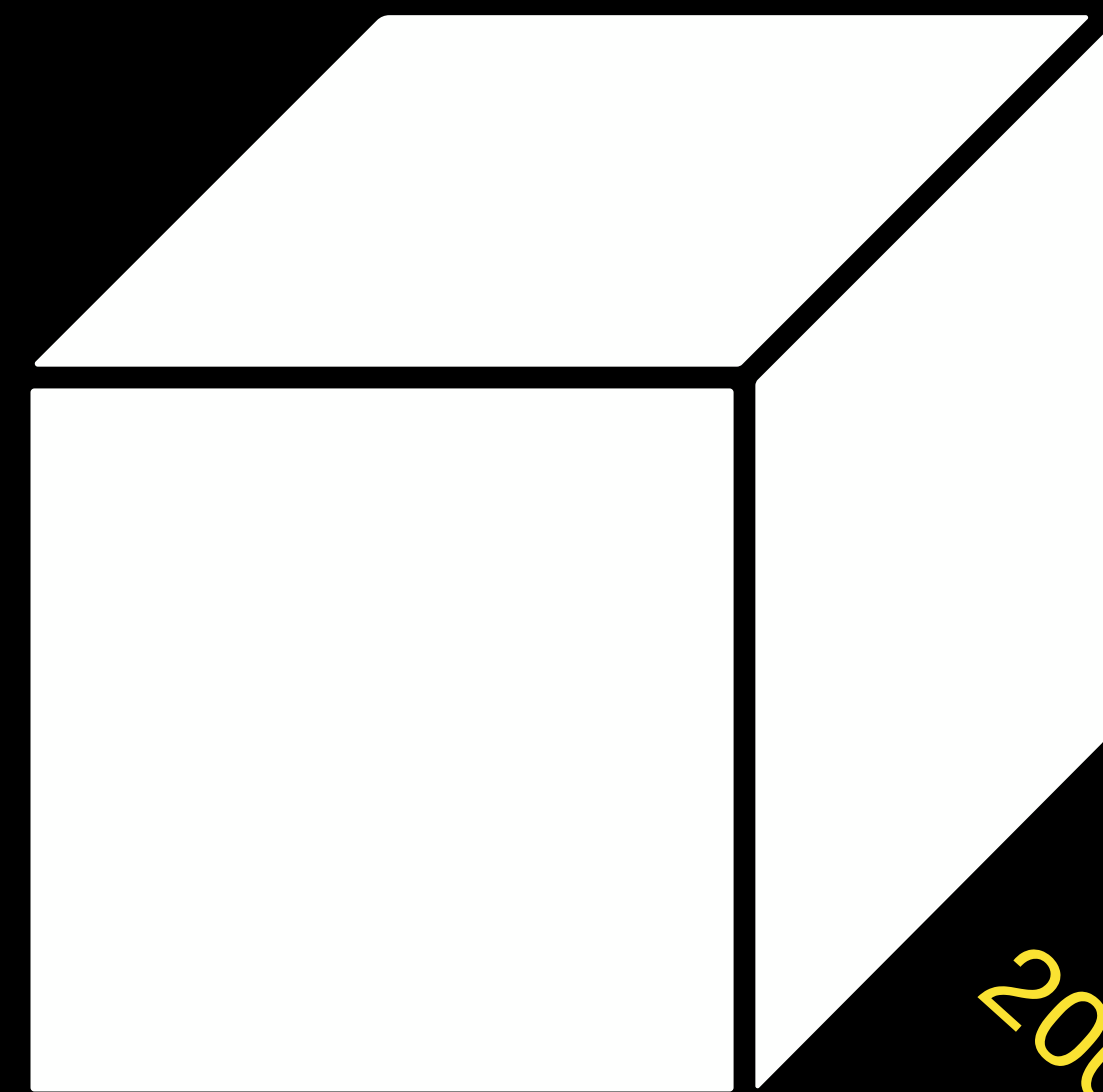
Reynolds nonlinearity

compute disk

Data dumps at 10x per correlation time for averages across the statistically steady state

Read out domain into HDF5 chunks, grouped by m nodes writing each chunk

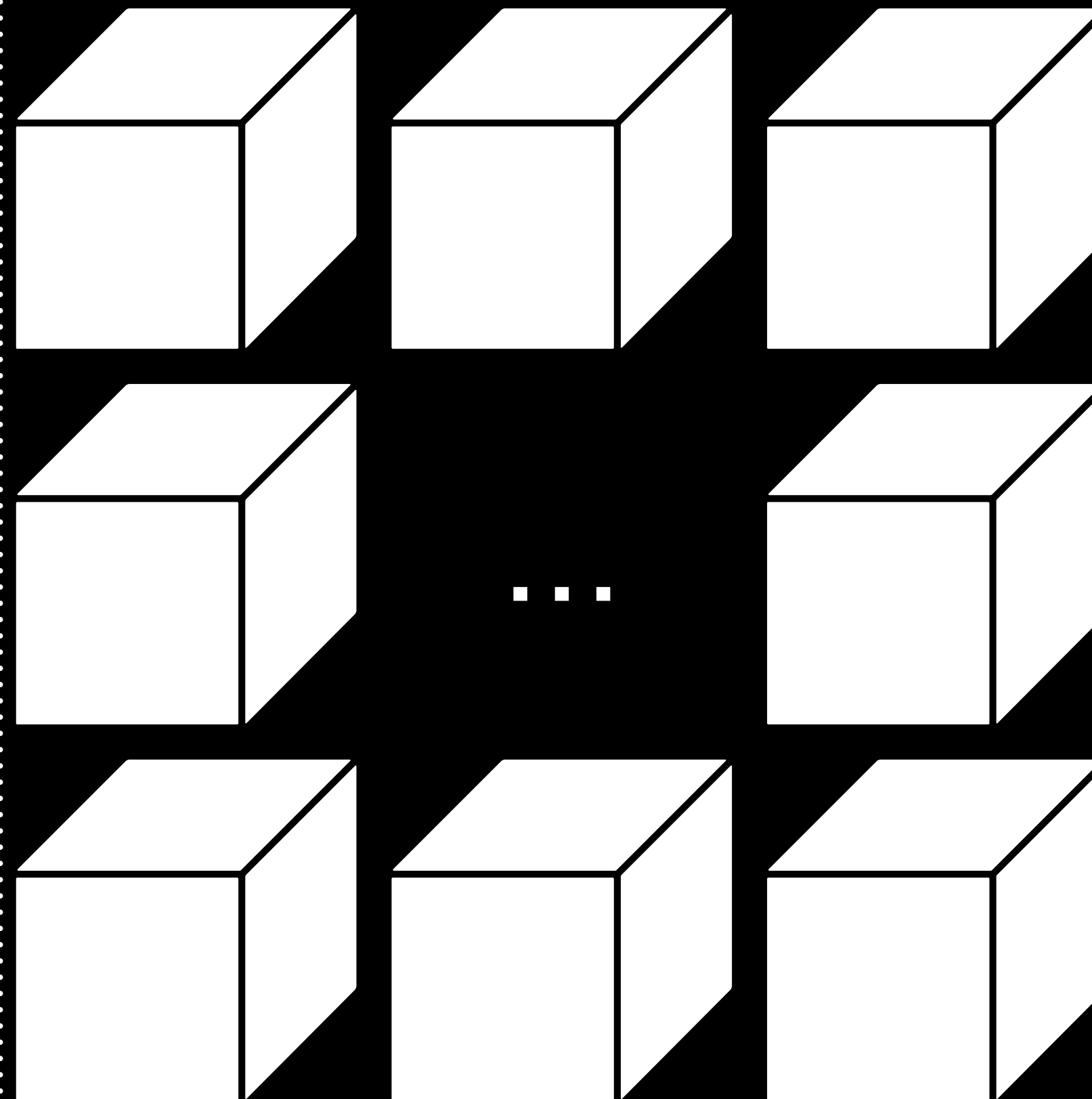
Merge into single HDF5 file in post-processing (C++ mpi tool)



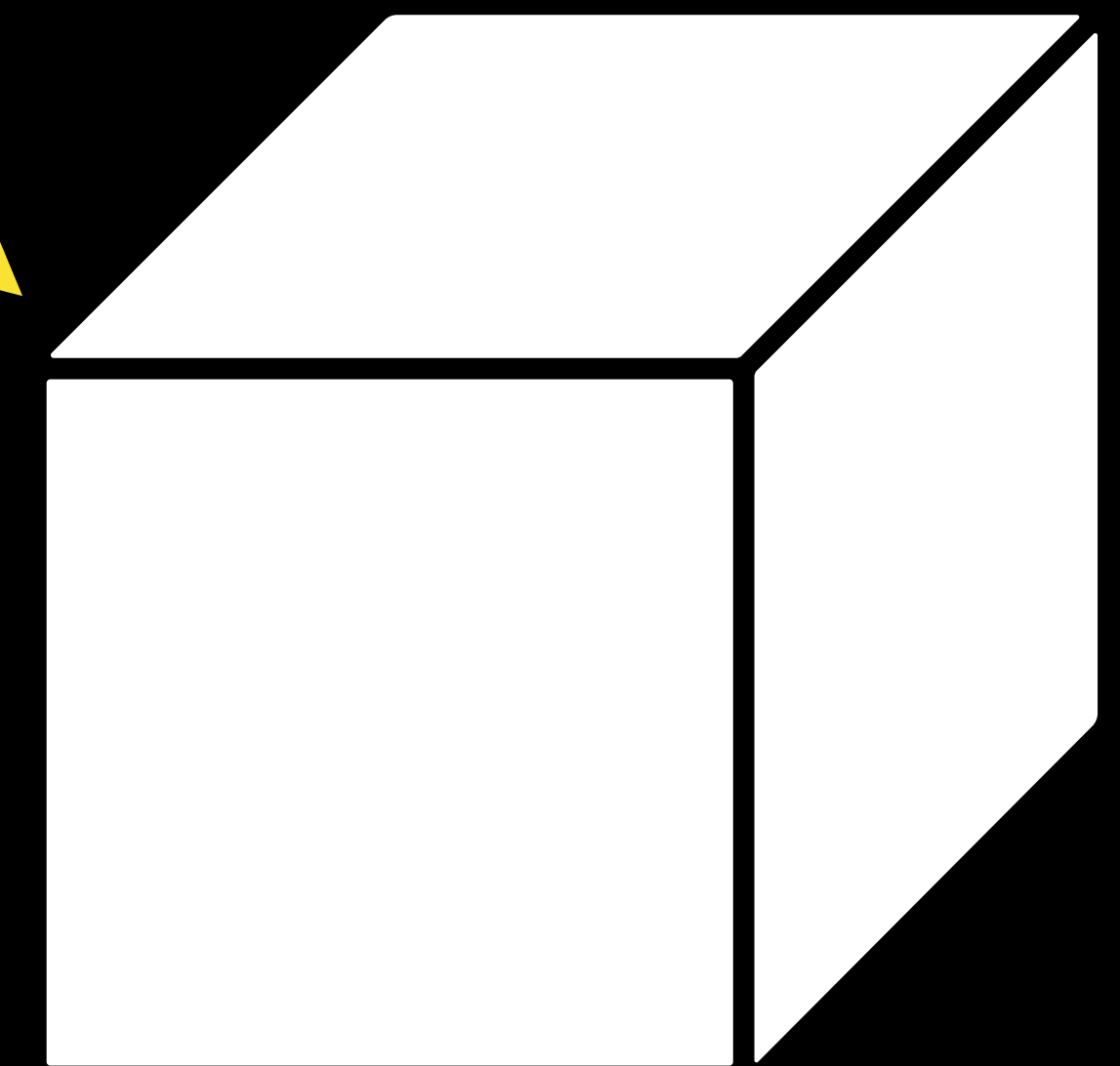
$10,080^3$

200 GB/sec

Almost max. bandwidth for SuperMUC-NG scratch



180 sec to write a 28 TB checkpoint file

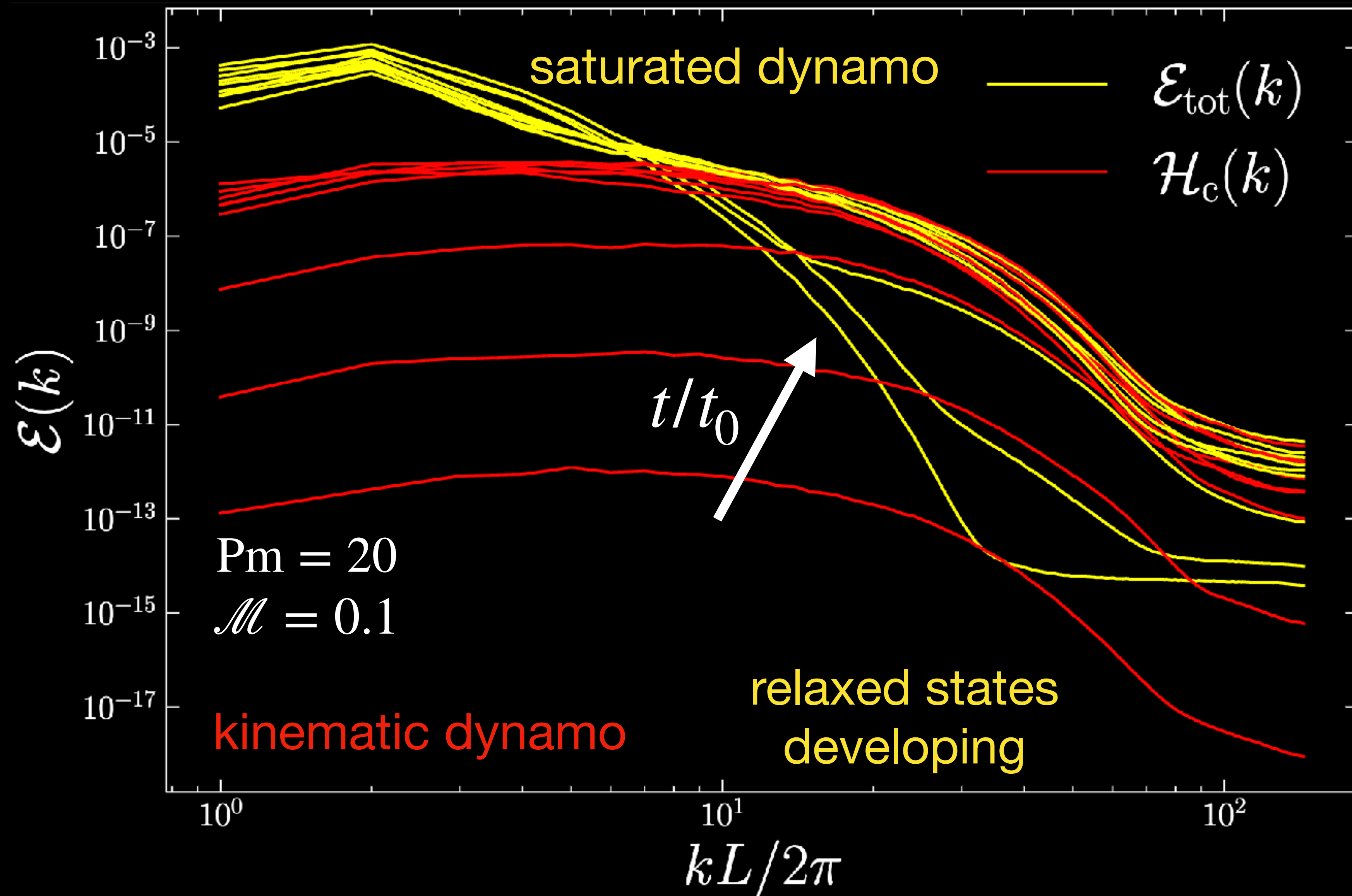


$10,080^3$

C++ & FFTW backend
Power spectrum on 50,000 cores.
Spectra in ~ 5mins

Why? Second ingredient — cross helicity spectrum

Can we ~ conserve rugged invariants in space instead of just in time?



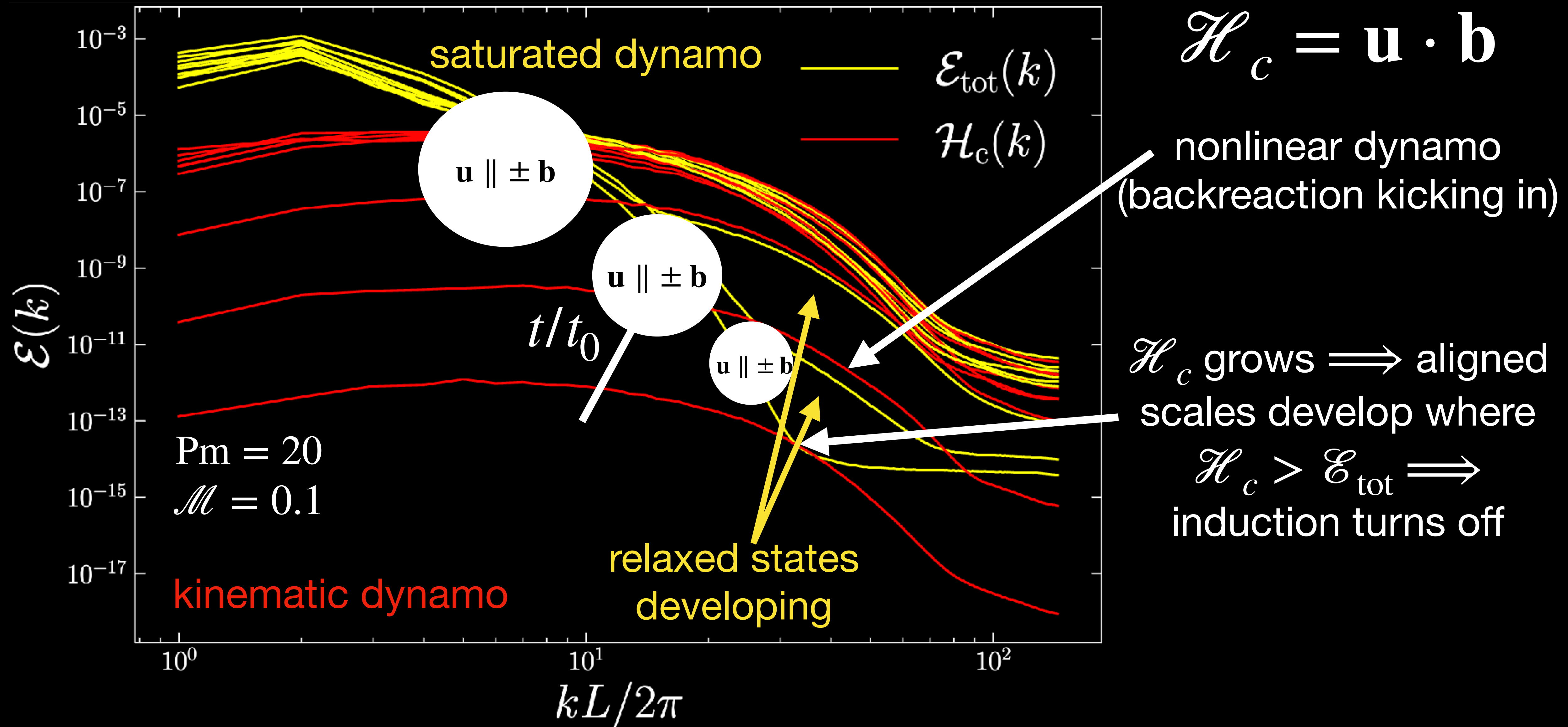
$$\mathcal{H}_c = \mathbf{u} \cdot \mathbf{b}$$

nonlinear dynamo
(backreaction kicking in)

$\mathcal{H}_c$ grows $\implies$ aligned
scales develop where
 $\mathcal{H}_c > \mathcal{E}_{\text{tot}} \implies$
induction turns off

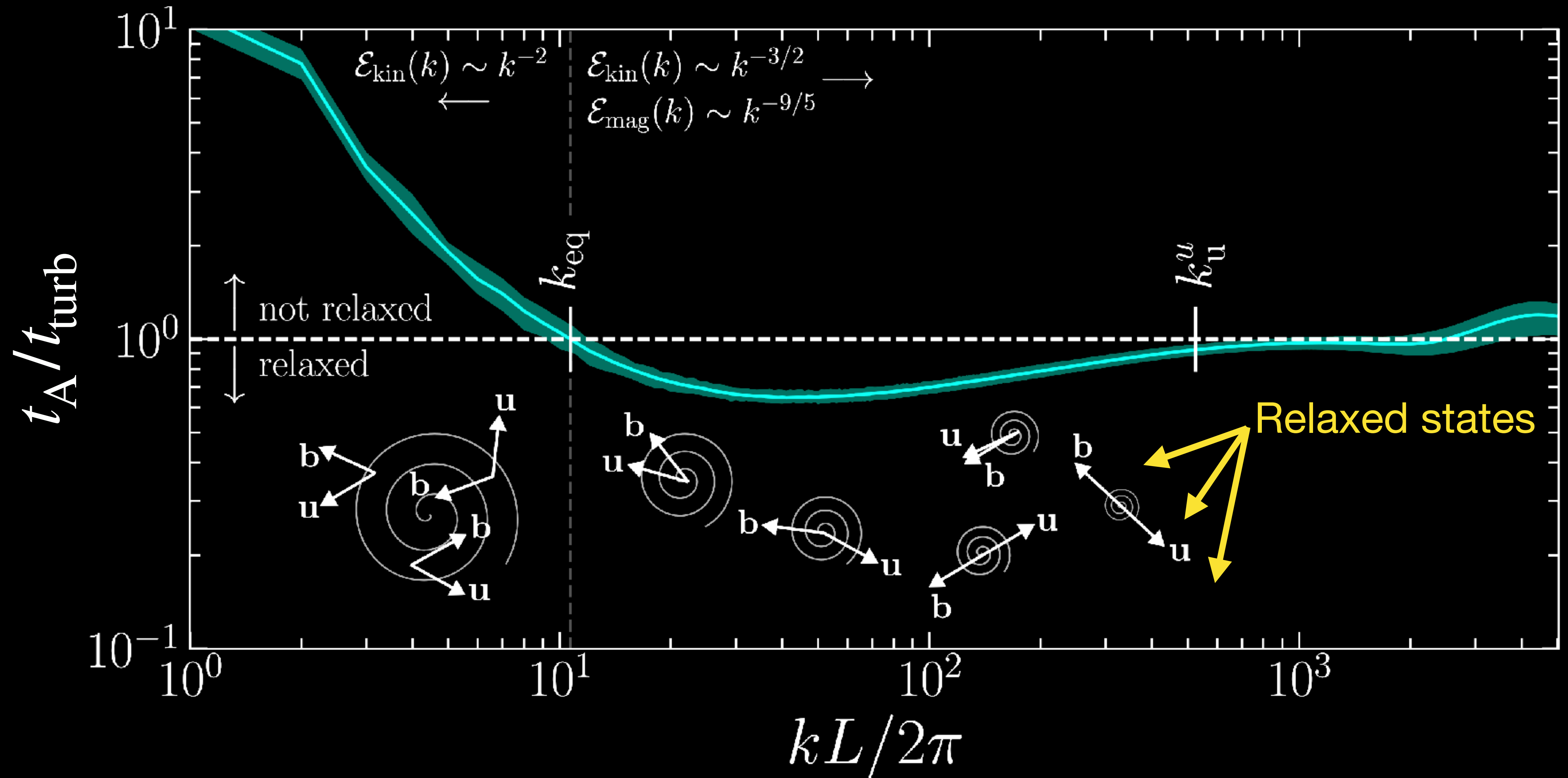
Why? Second ingredient — cross helicity spectrum

Can we ~ conserve rugged invariants in space instead of just in time?



Why? First ingredient — separation of timescales

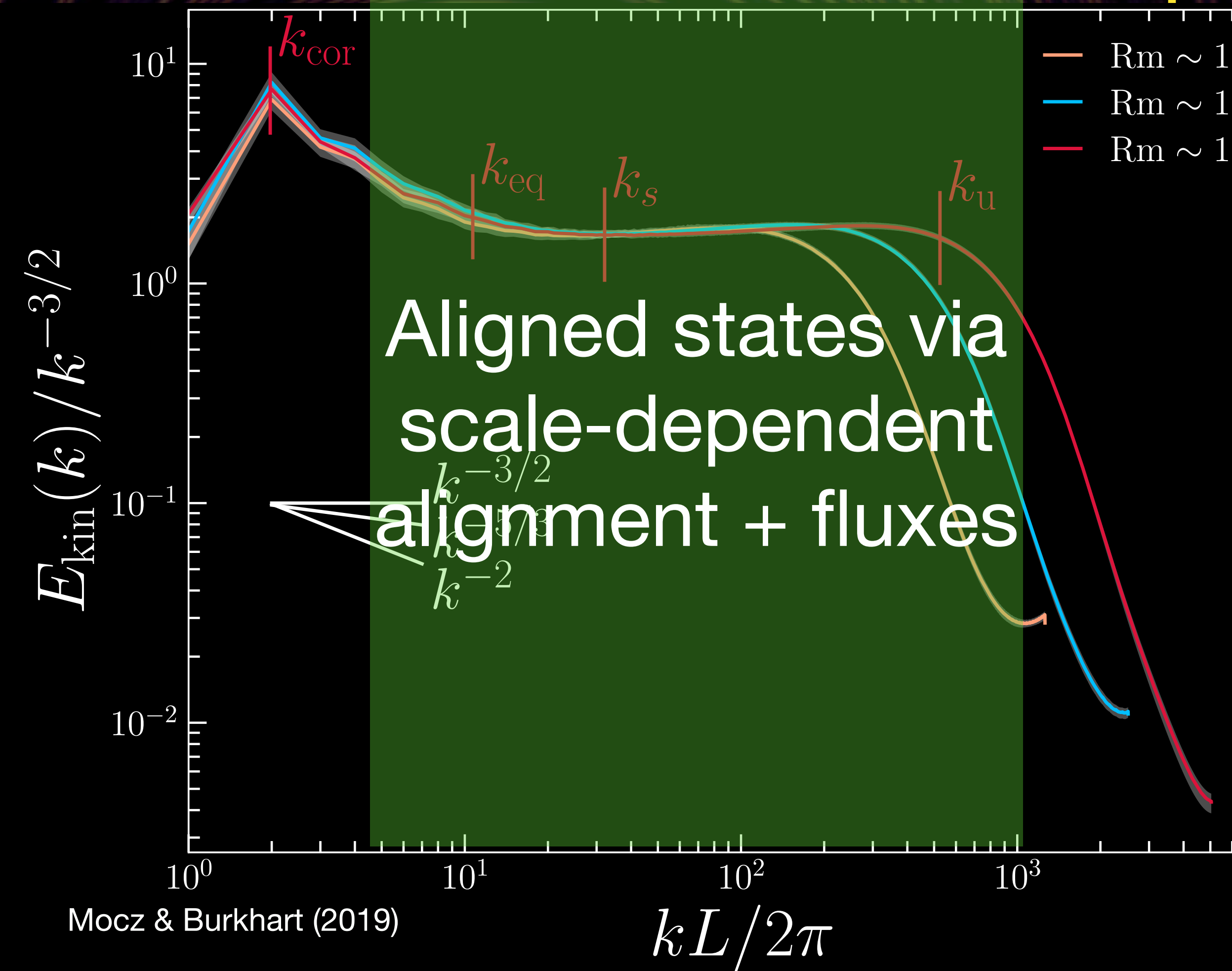
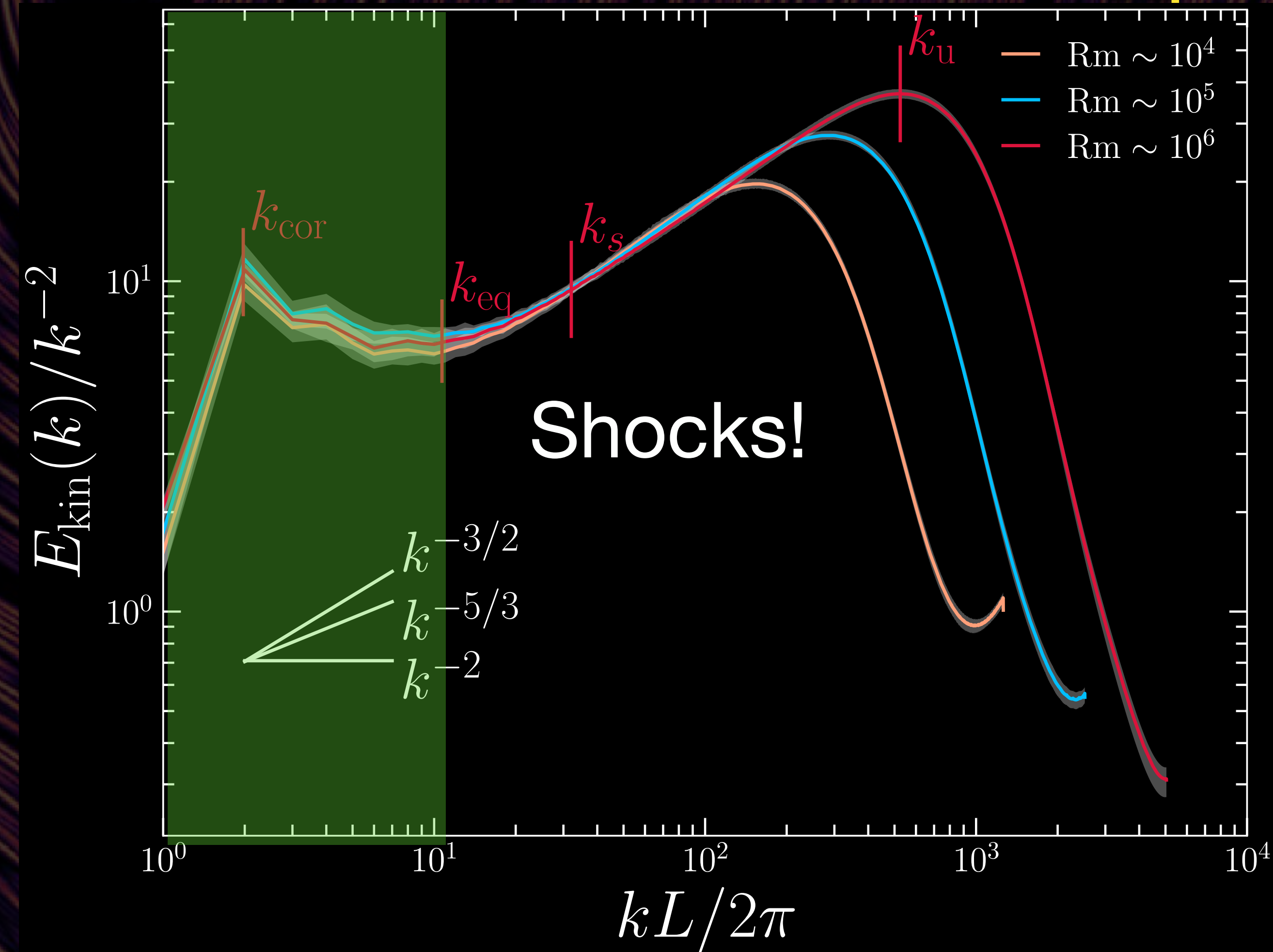
Can we align faster than the turbulence can perturb magnetic field out of alignment?



Velocity spectrum

$$\mathcal{E}(k) \sim k^{-2}, k \leq k_{\text{eq}}$$

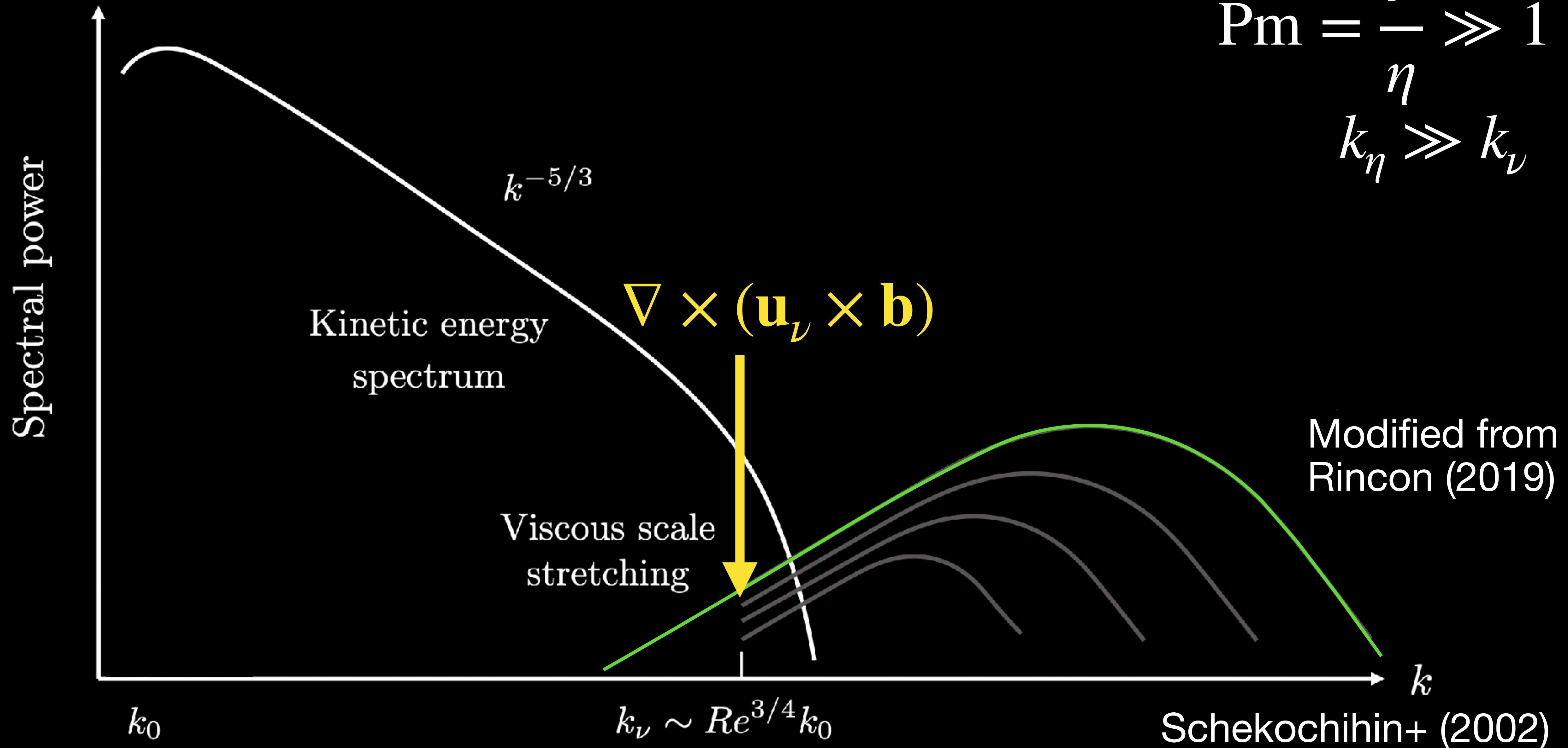
$$\mathcal{E}(k) \sim k^{-3/2}, k > k_{\text{eq}}$$



The dynamical saturation of the dynamo

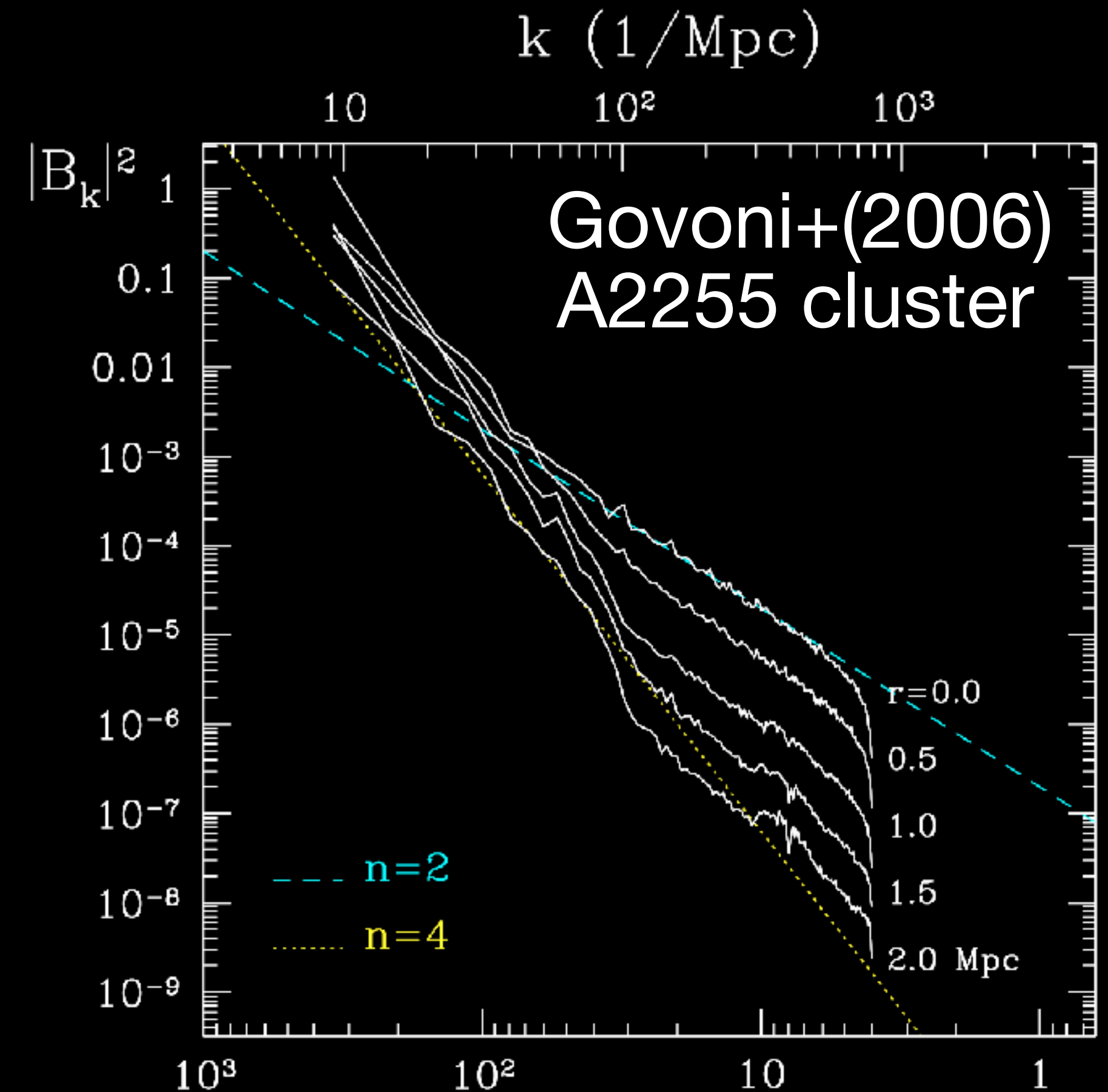
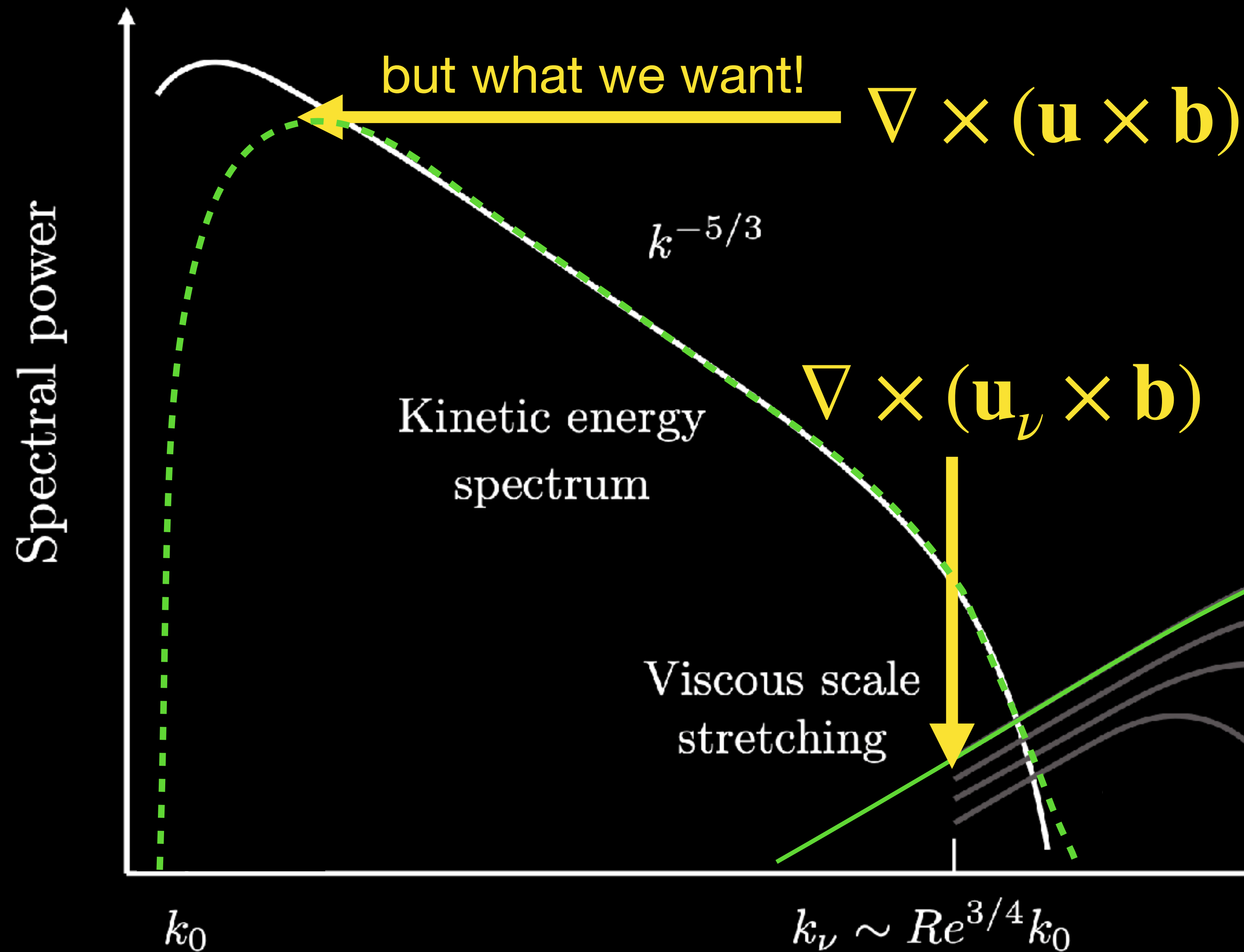
Saturation through alignment

$$\text{Pm} = \frac{\nu}{\eta} \gg 1$$
$$k_\eta \gg k_\nu$$



The dynamical saturation of the dynamo

Saturation through alignment



$\Lambda(\text{kpc})$

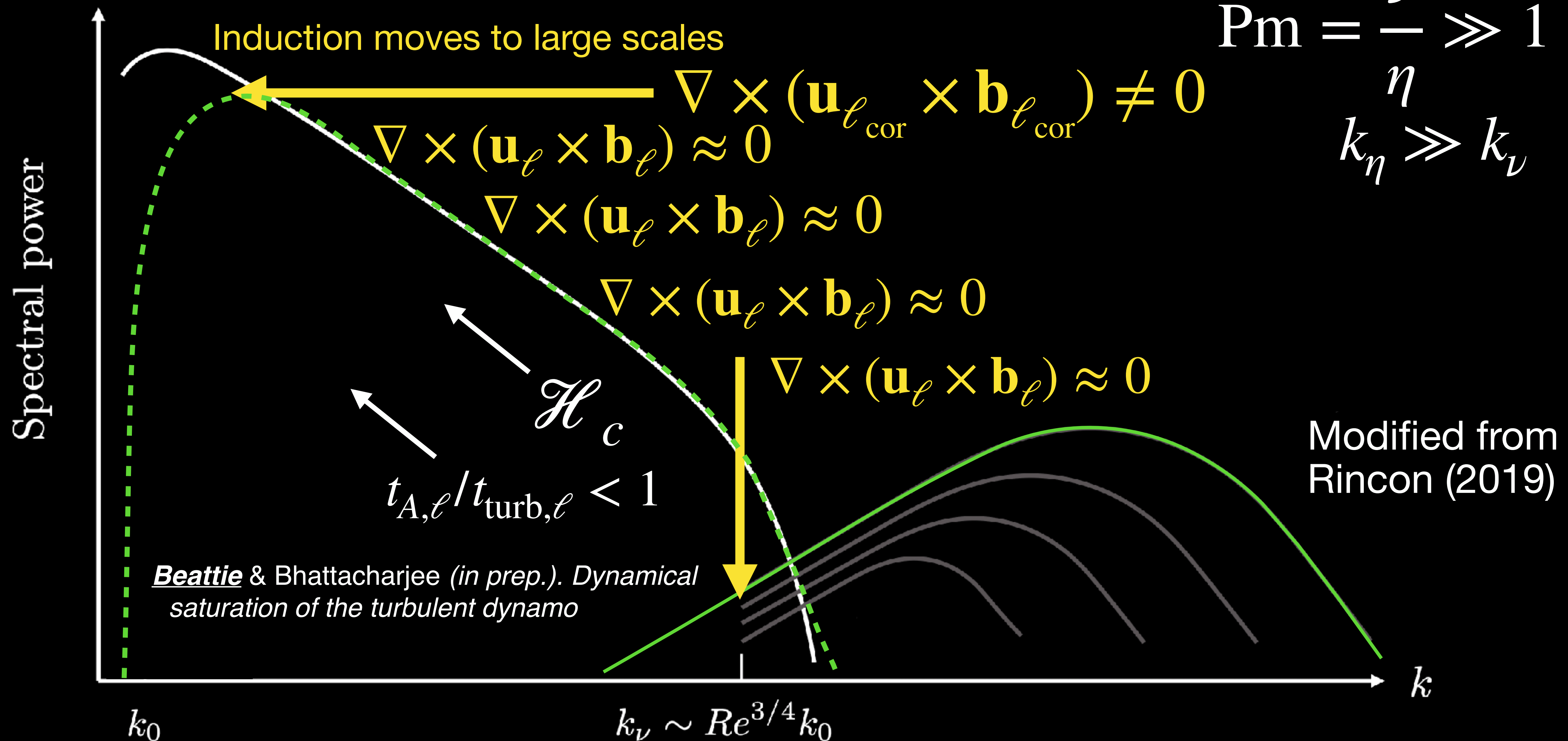
Schekochihin+ (2002)

The dynamical saturation of the dynamo

Saturation through alignment

$$\text{Pm} = \frac{\nu}{\eta} \gg 1$$

$$k_\eta \gg k_\nu$$



Inevitability of the turbulent dynamo

Saturation through alignment

$$\mathbf{k}' \xrightarrow{\mathbf{k}''} \mathbf{k}'''$$

Alignment implies a perfect balance between dynamo and cascade energy fluxes

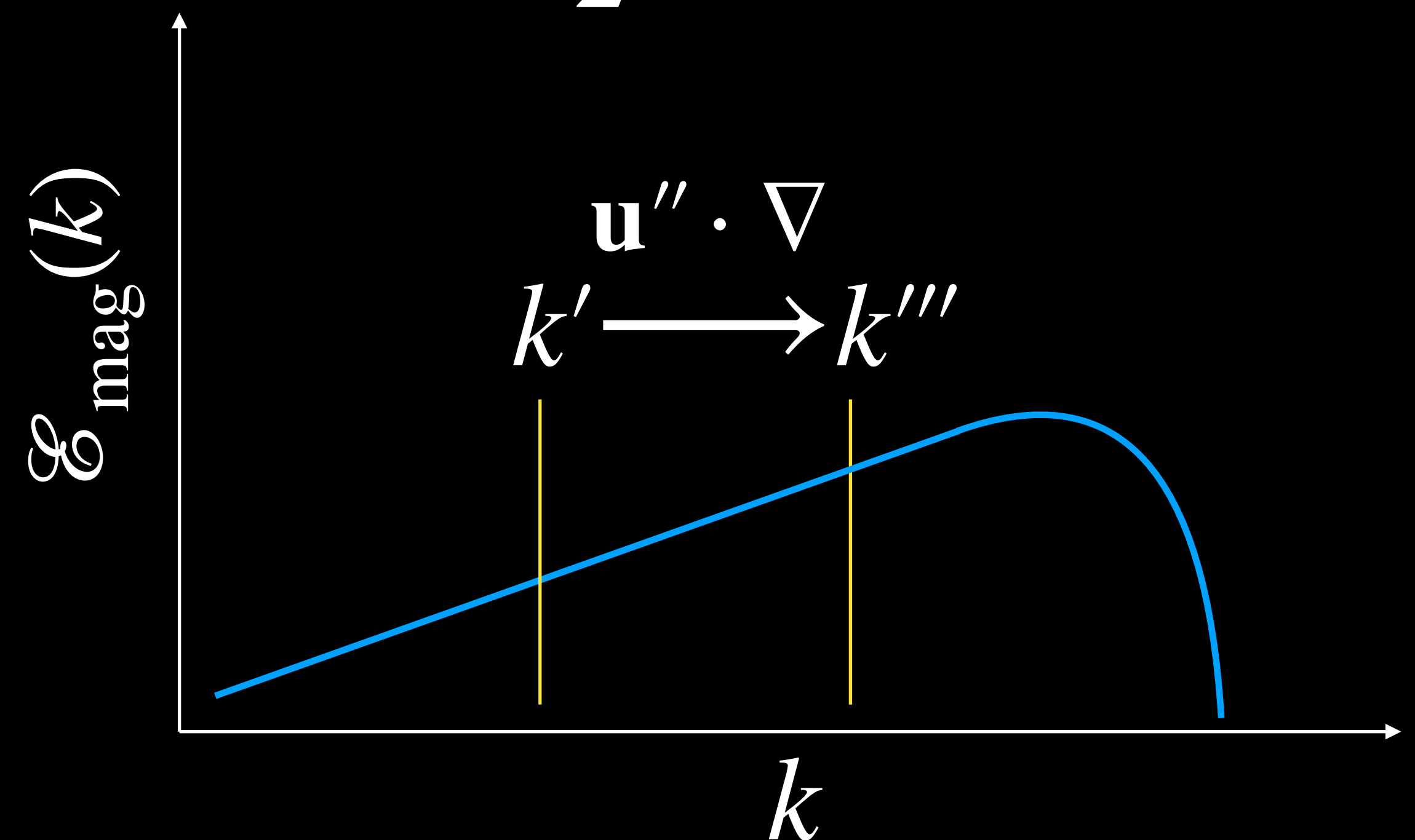
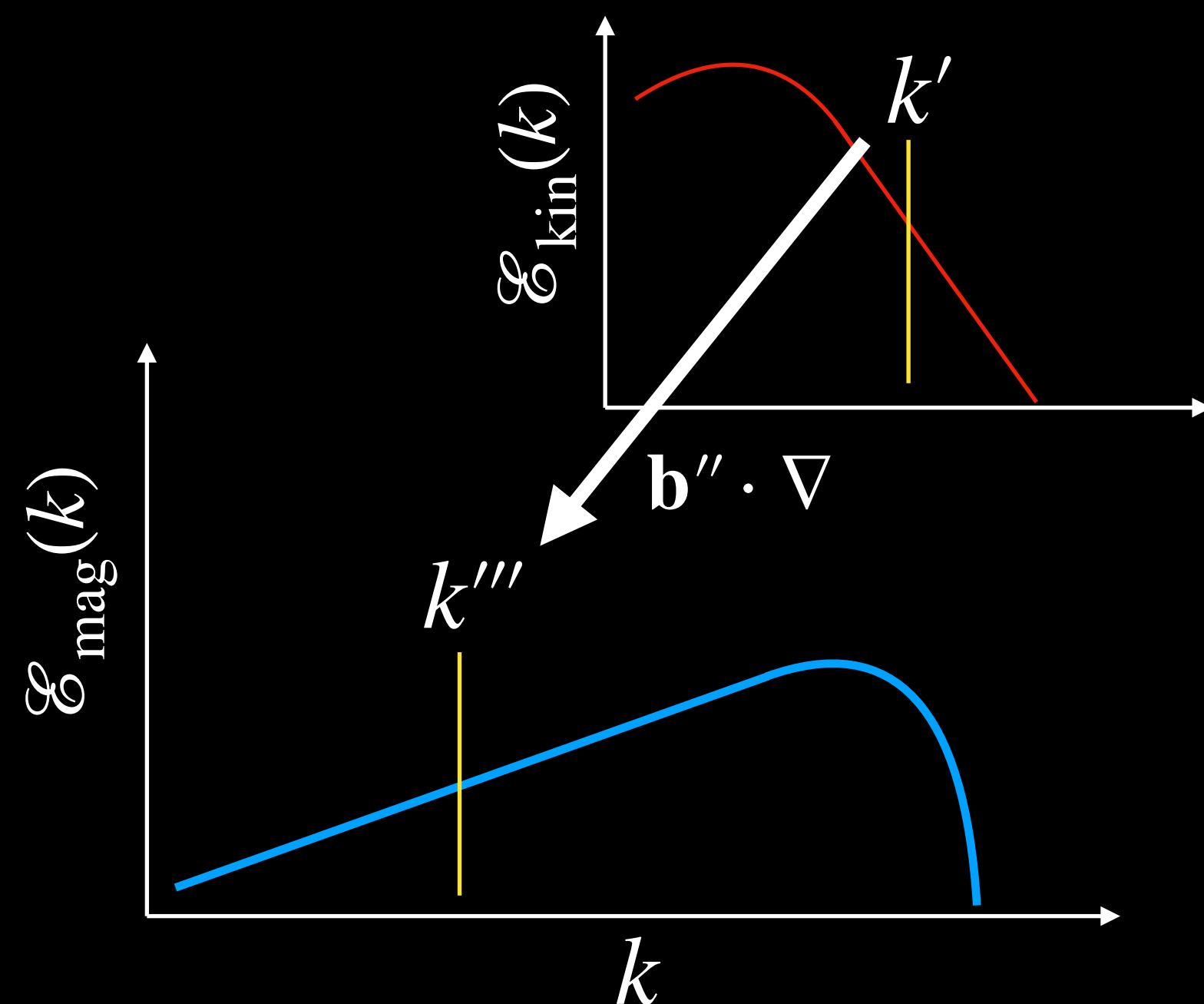
$$\nabla \times (\mathbf{u}_\ell \times \mathbf{b}_\ell) \approx 0 \implies$$

$$\underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}'}_{\text{kinetic to magnetic energy transfer}}$$

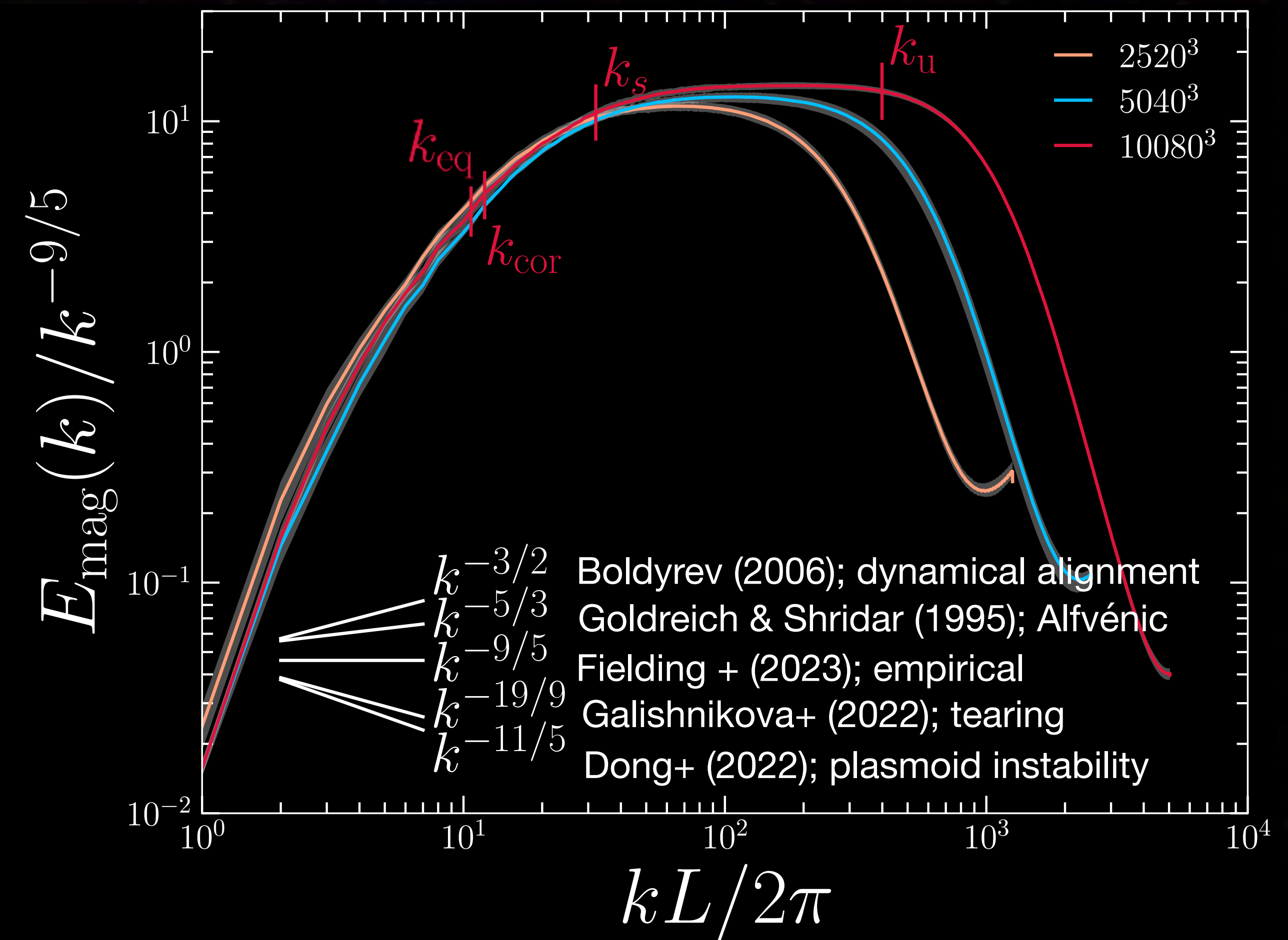
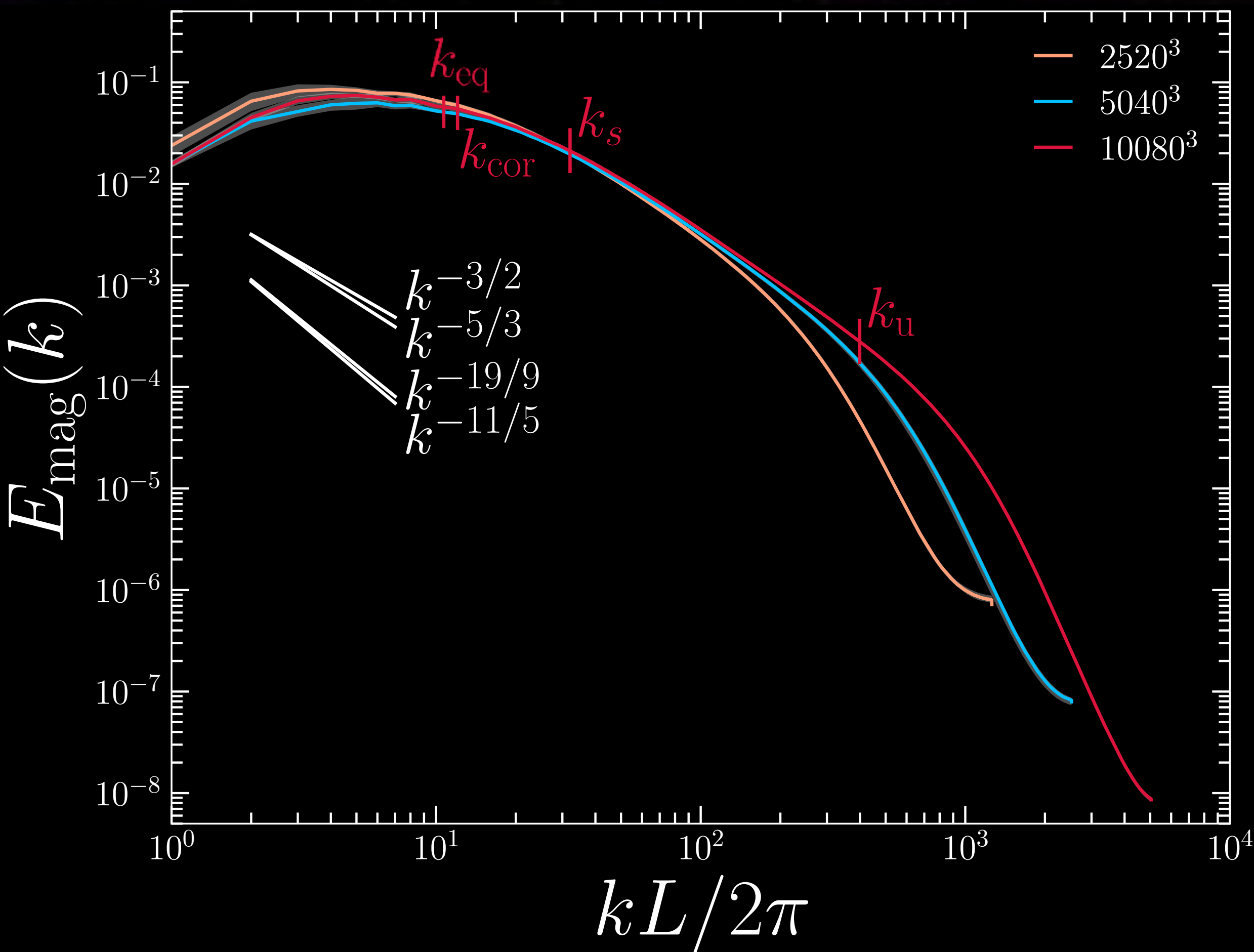
kinetic to magnetic energy transfer

magnetic cascade terms

$$= \mathbf{b}''' \otimes \mathbf{u}'' : \nabla \otimes \mathbf{b}' + \frac{1}{2} \mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}'') \mathbb{I}$$



Magnetic spectrum in saturated state



Beattie+(2024). Magnetized compressible turbulence with a fluctuation dynamo and Reynolds numbers over a million [arXiv:2405.16626](https://arxiv.org/abs/2405.16626)

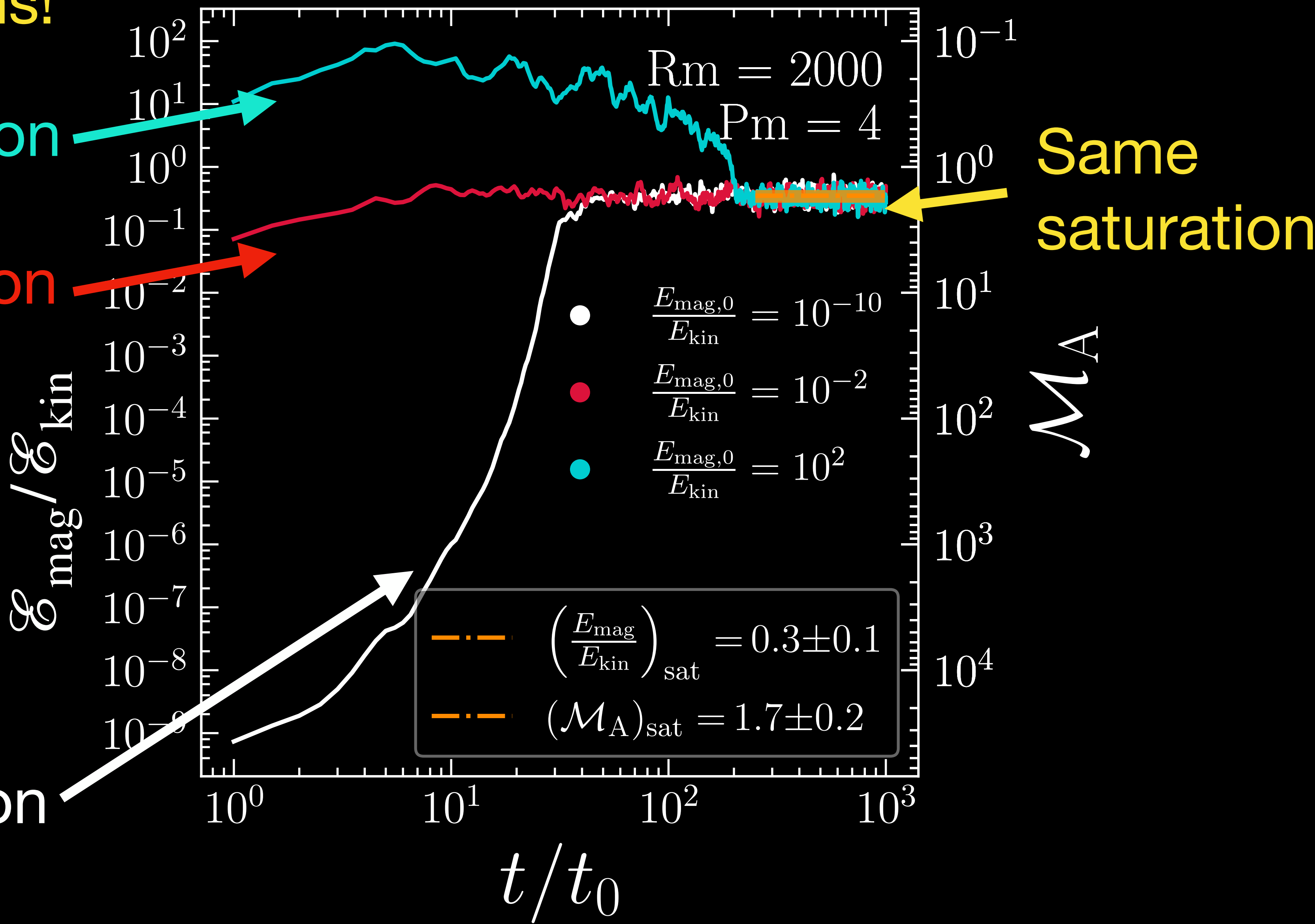
Worried about magnetic ICs? Turbulent dynamo takes care of initial magnetic field conditions!

Decay into the saturation

Initialise in the saturation

$$\mathcal{M}_A = \sqrt{\frac{E_{\text{kin}}}{E_{\text{mag}}}}$$

Grow into the saturation



Magnetic Relaxation — main idea?

Searching to weaken the nonlinearities

Define constraint equation based on quadratic (ideal) MHD invariants

$$H_m = \langle \mathbf{a} \cdot \mathbf{b} \rangle$$

$$H_c = \langle \mathbf{u} \cdot \mathbf{b} \rangle$$

total energy

cross helicity

$$\mathcal{E} - \lambda_1 H_m - \lambda_2 H_c = \text{const.}$$

magnetic helicity

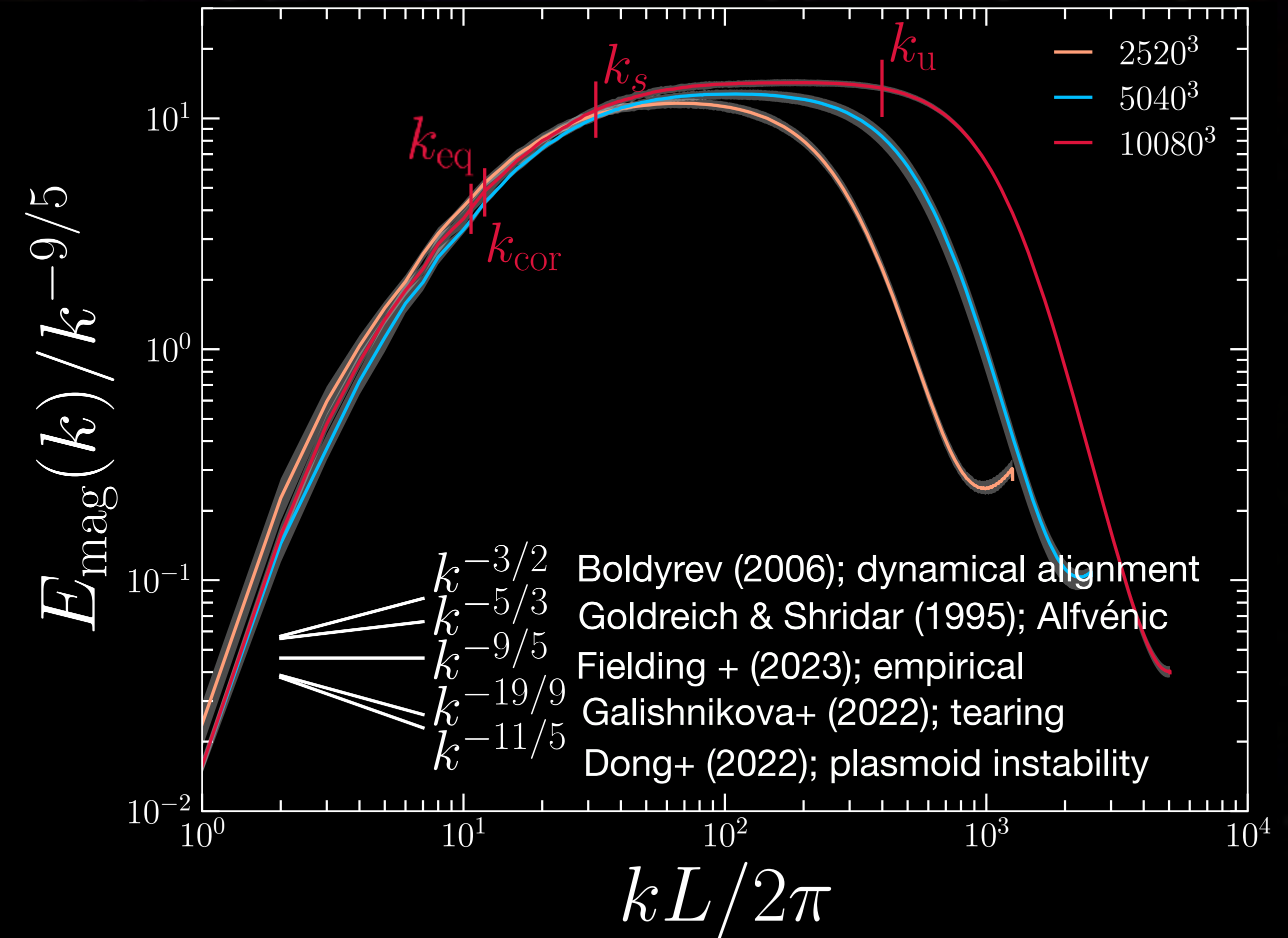
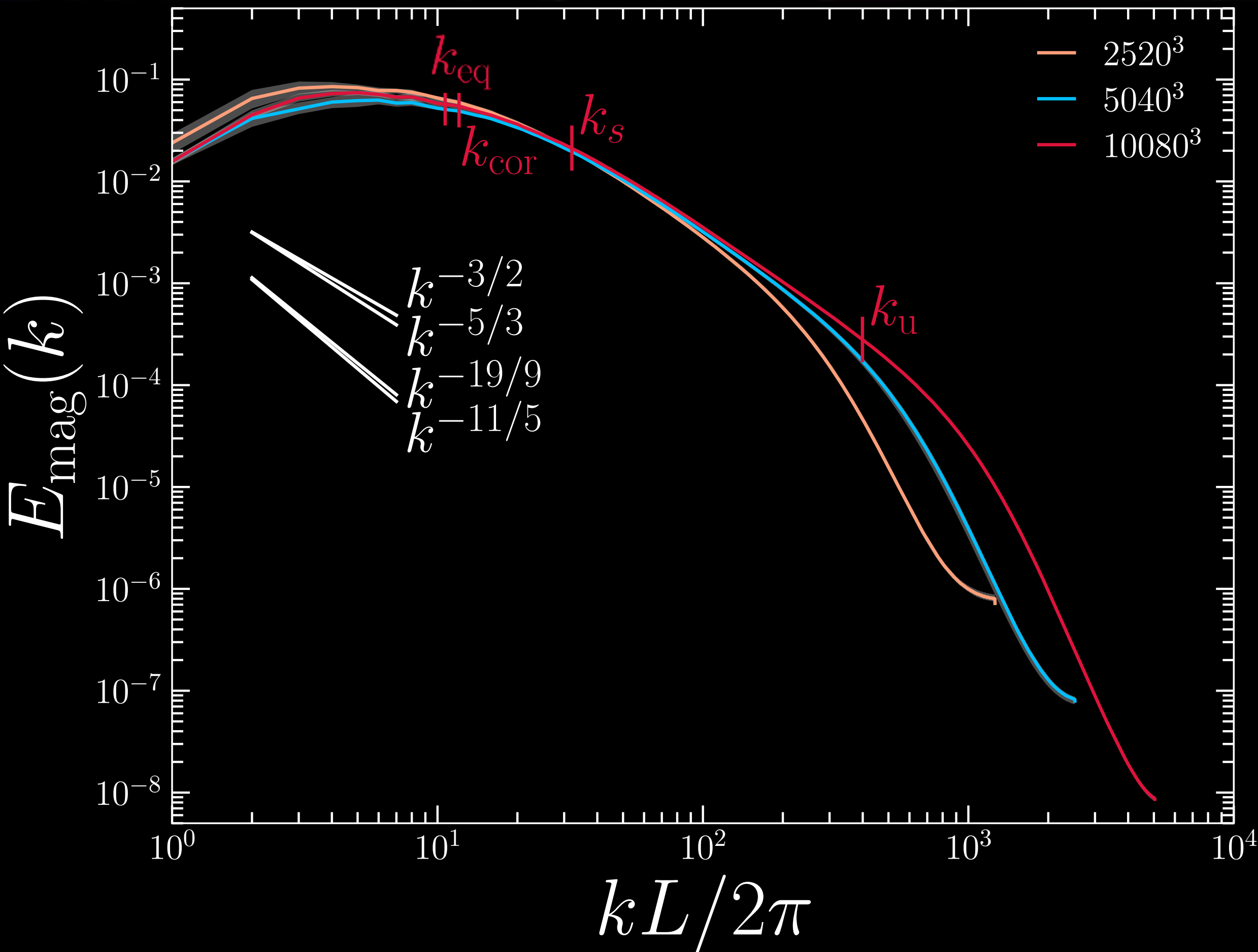
Use variational principle on magnetic energy eq.,

$$\delta \left(\mathcal{E} - \lambda_1 H_m - \lambda_2 H_c \right) = 0,$$

Minimize to find minimum energy state.

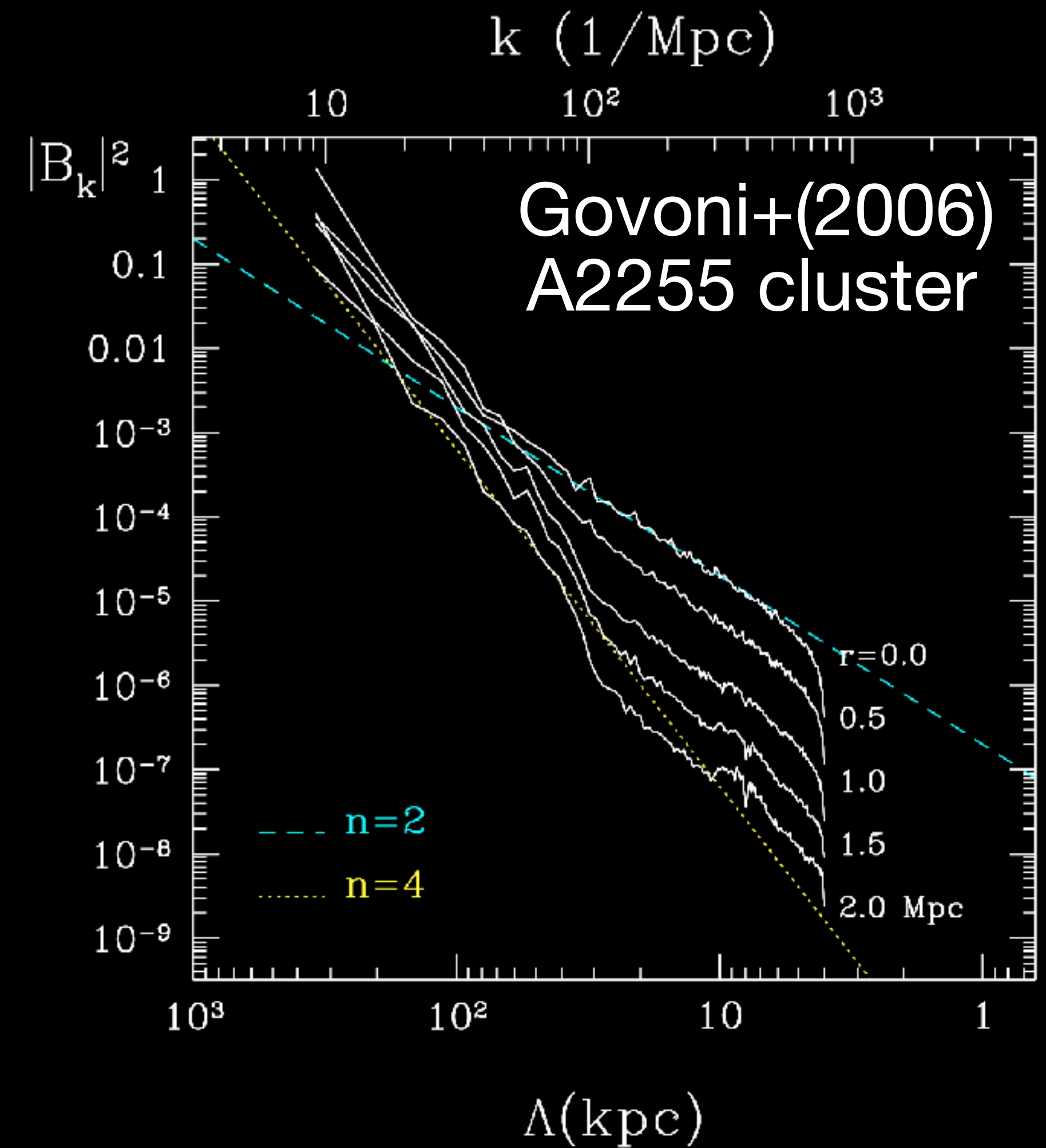
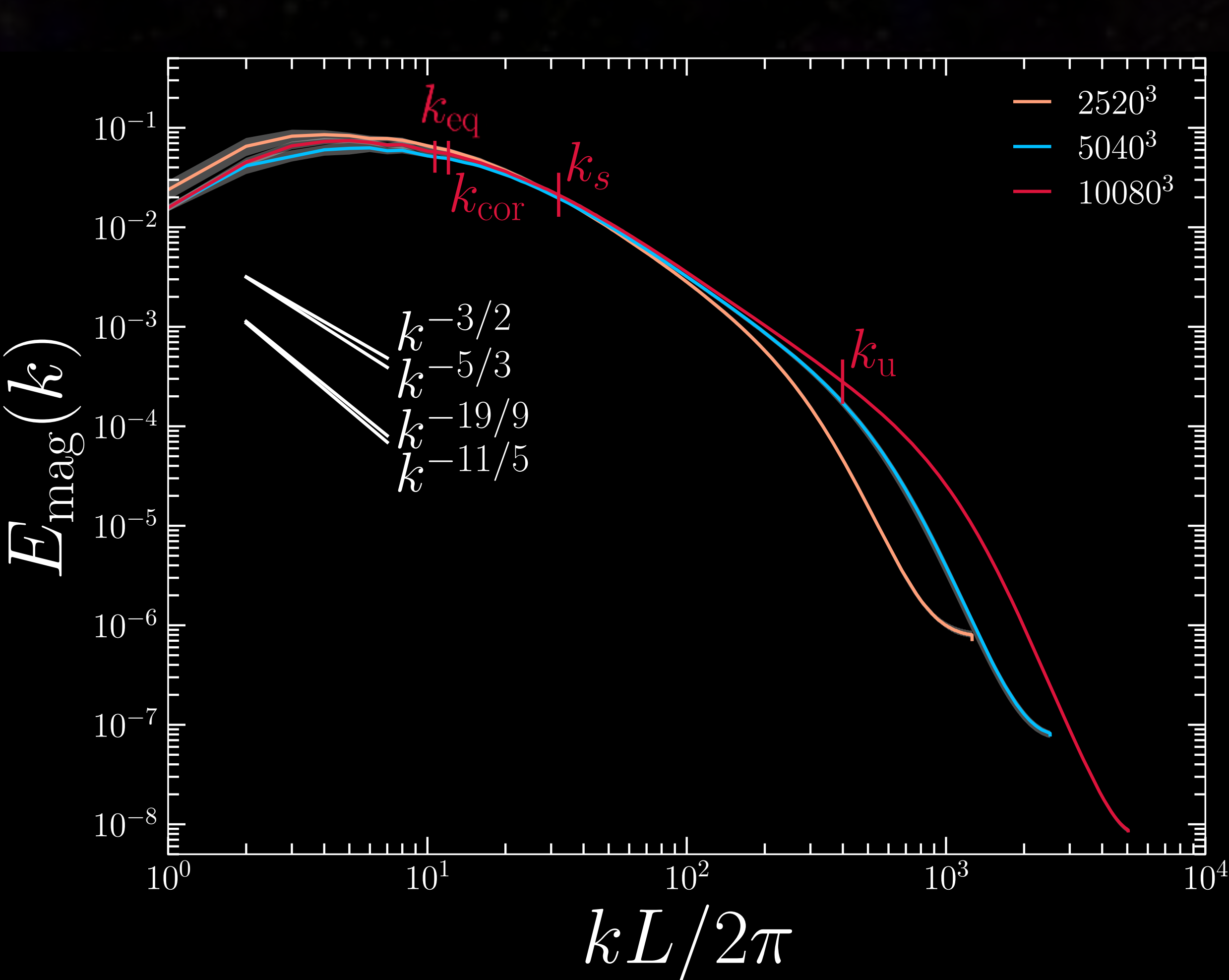
$$\mathbf{u} = \lambda_1 \mathbf{b}, \quad \mathbf{j} = 2\lambda_2 \mathbf{b} + \lambda_1 \boldsymbol{\omega}$$

Magnetic cascade in saturated dynamo



Beattie+(2024). Magnetized compressible turbulence with a fluctuation dynamo and Reynolds numbers over a million [arXiv:2405.16626](https://arxiv.org/abs/2405.16626)

Magnetic cascade in saturated dynamo



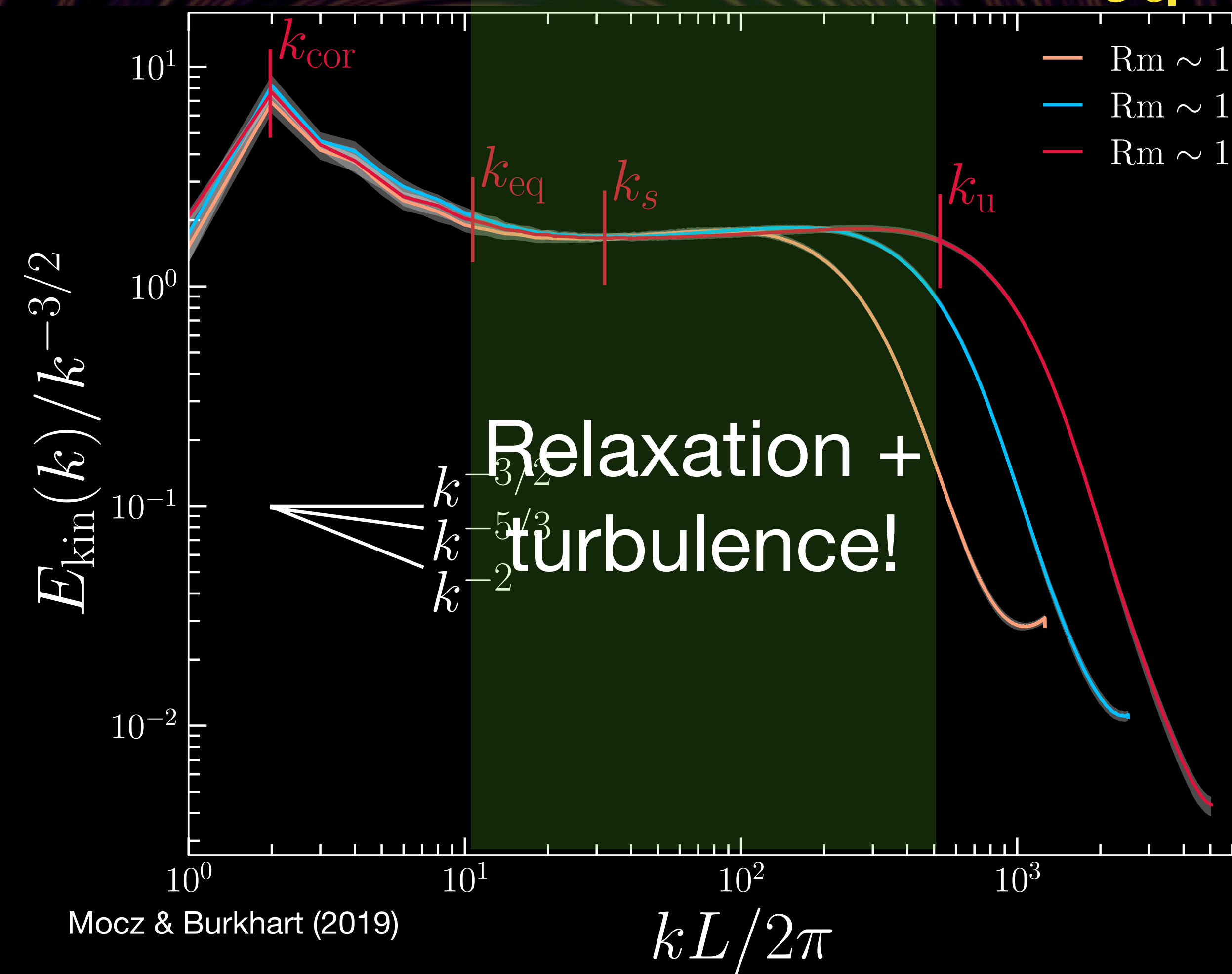
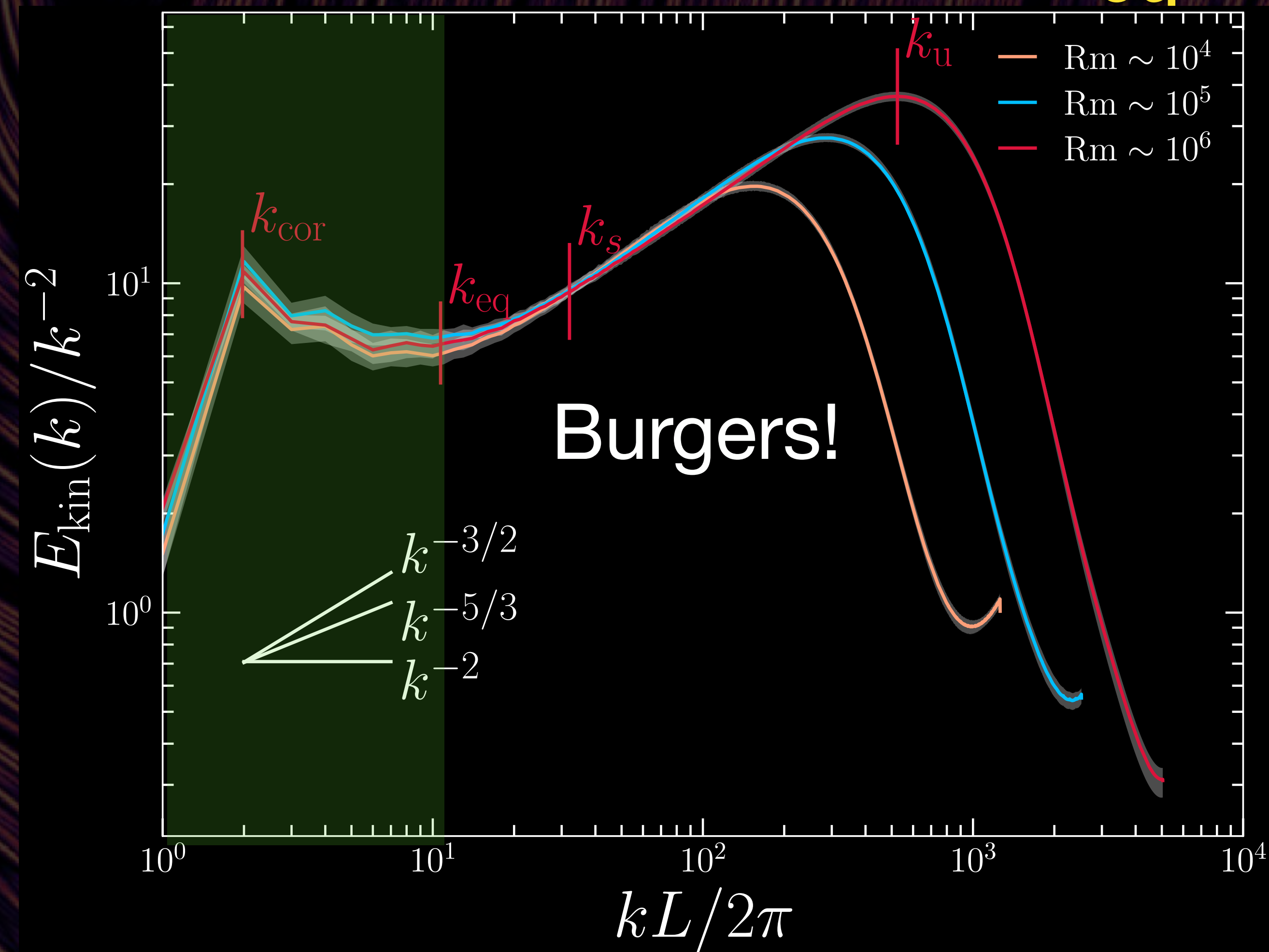
Beattie+(2024). Magnetized compressible turbulence with a fluctuation dynamo and Reynolds numbers over a million [arXiv:2405.16626](https://arxiv.org/abs/2405.16626)

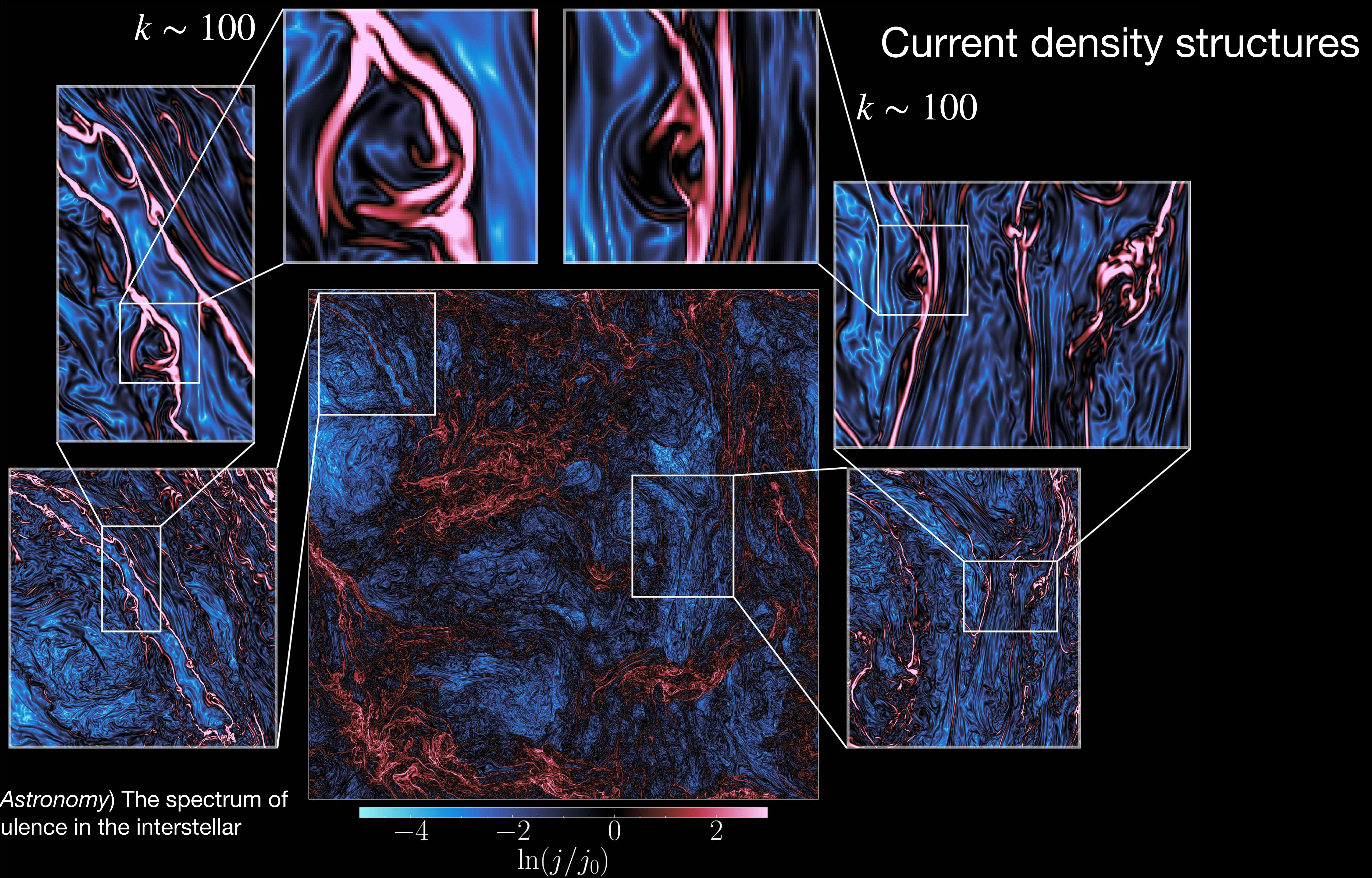
- **HIGH-RES: 10,080³ (80.0Mcore-h, 148,240cores)**
- **3.45PB of data products**
- **Factor of 4 higher linear grid resolution than IllustrisTNG**

The supersonic energy cascade

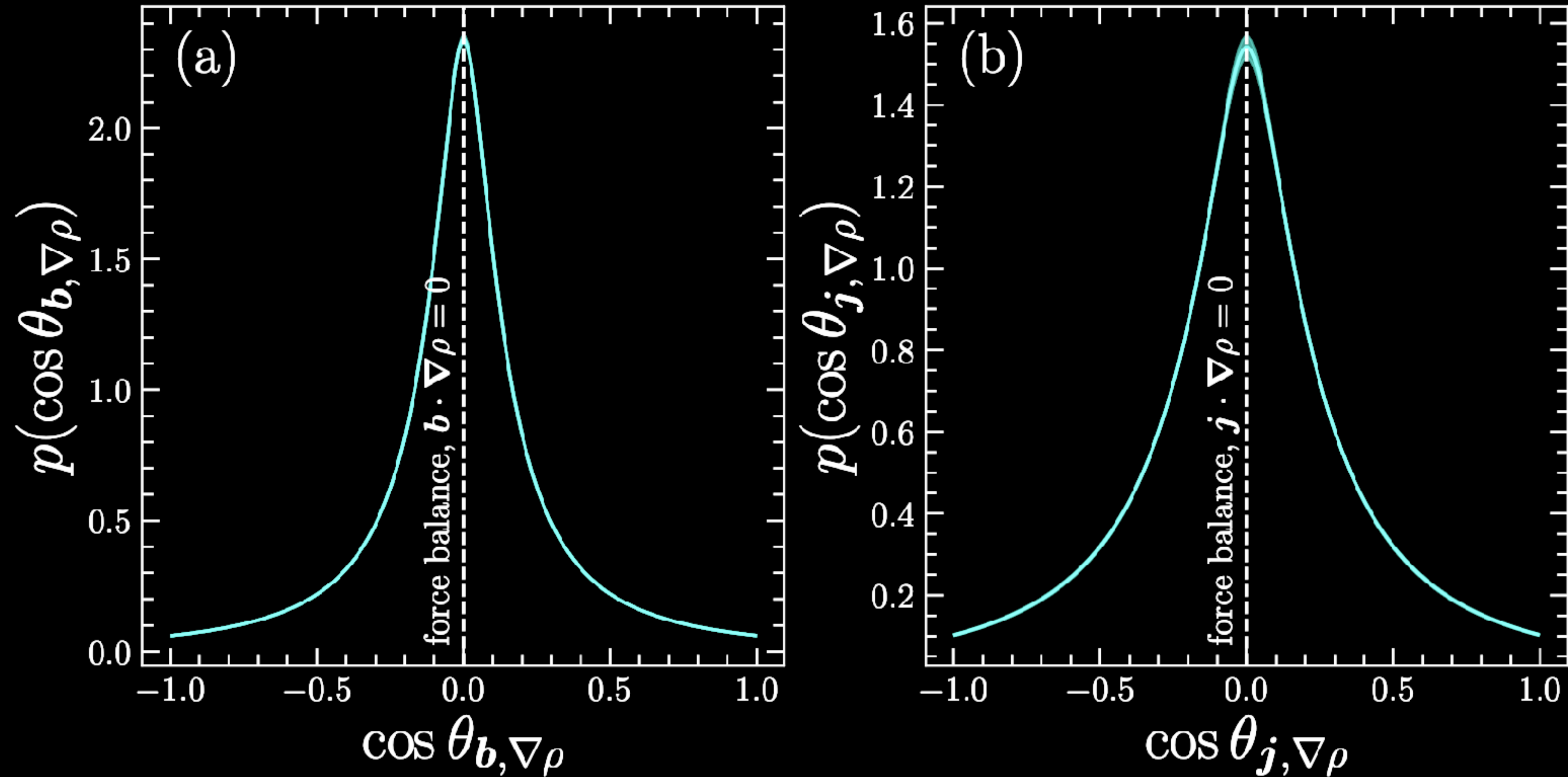
$$\mathcal{E}(k) \sim k^{-2}, k \leq k_{\text{eq}}$$

$$\mathcal{E}(k) \sim k^{-3/2}, k > k_{\text{eq}}$$





Compressibility changes the nature of the current structure



$$\mathbf{j} \times \mathbf{b} \propto \nabla \rho$$

Current structures exist on orthogonal surfaces to $\nabla \rho$!

Magnetic Relaxation?

Searching to weaken the nonlinearities

Relaxing into a saturation

$$E_{\text{mag}} \gg E_{\text{kin}} \quad t = t_0$$

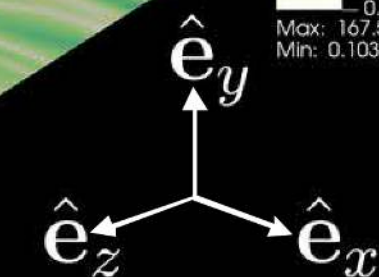
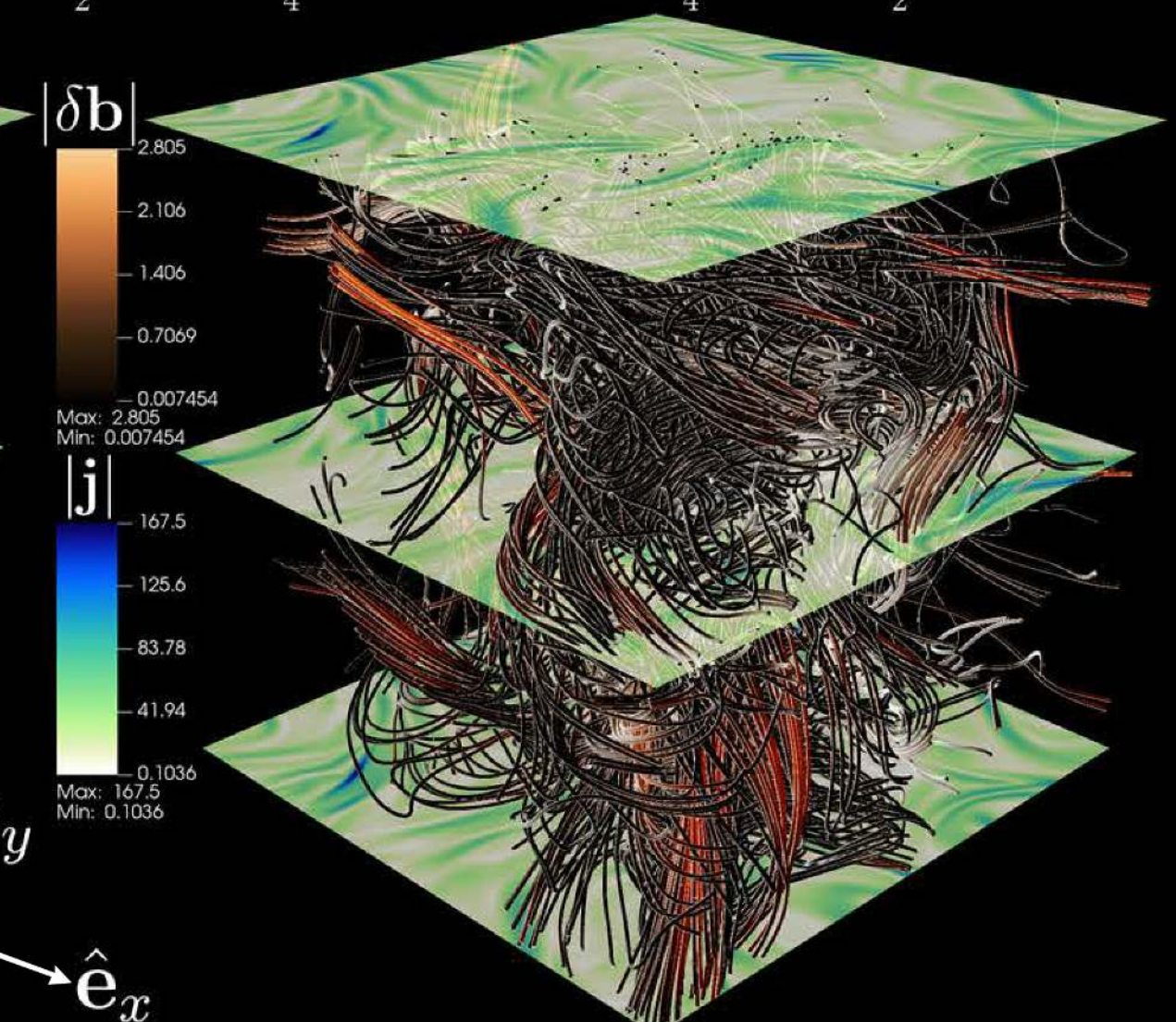
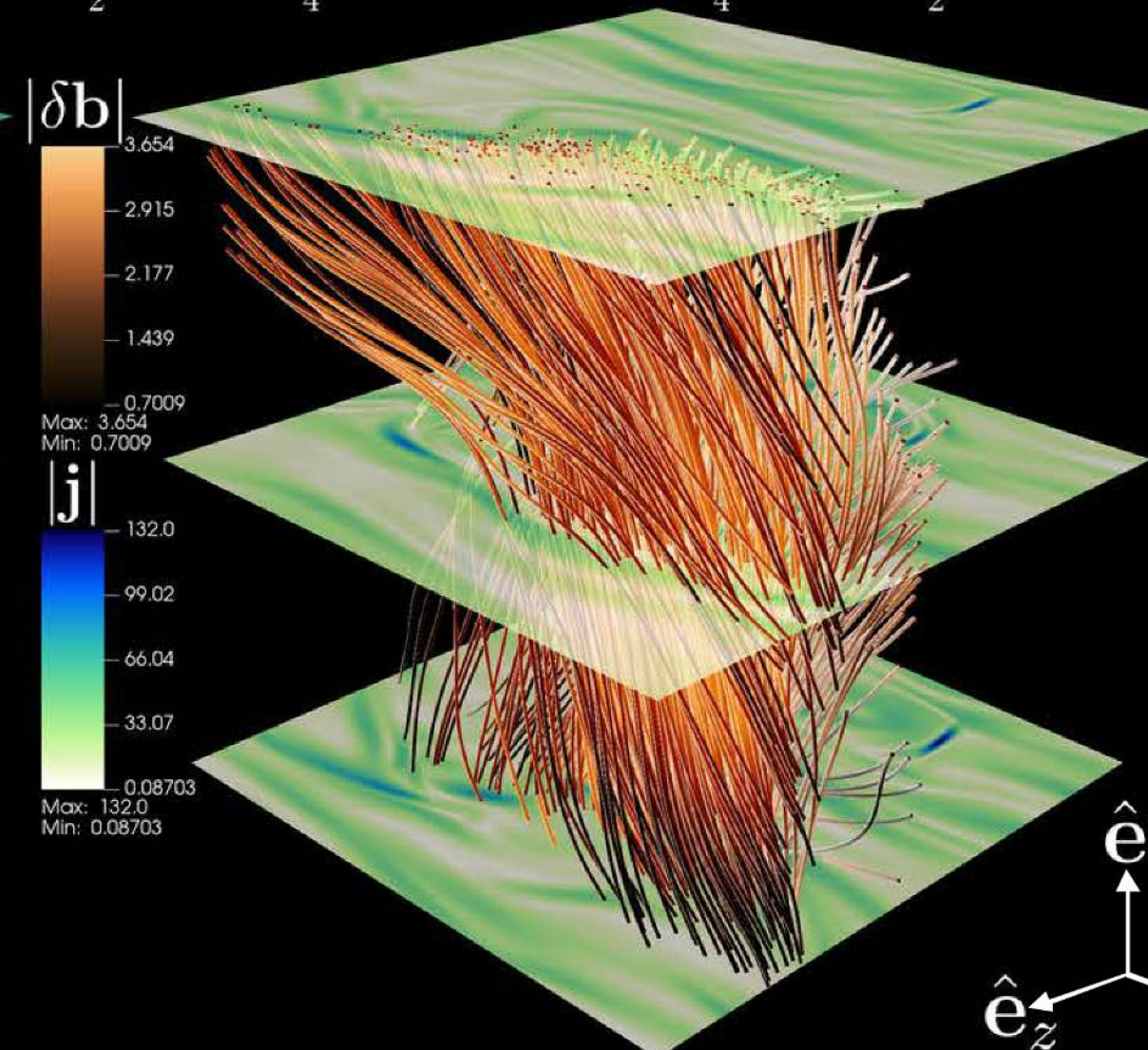
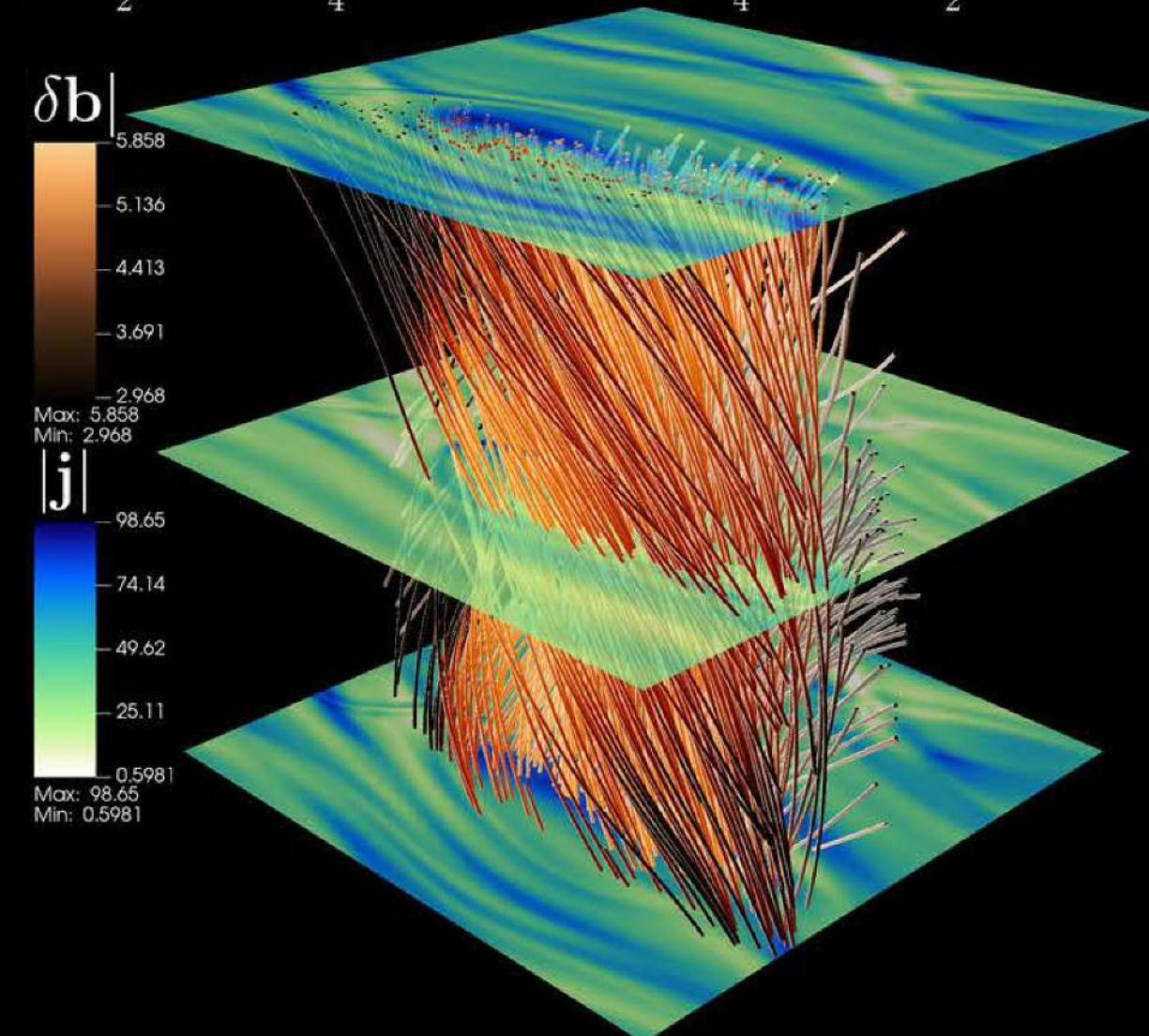
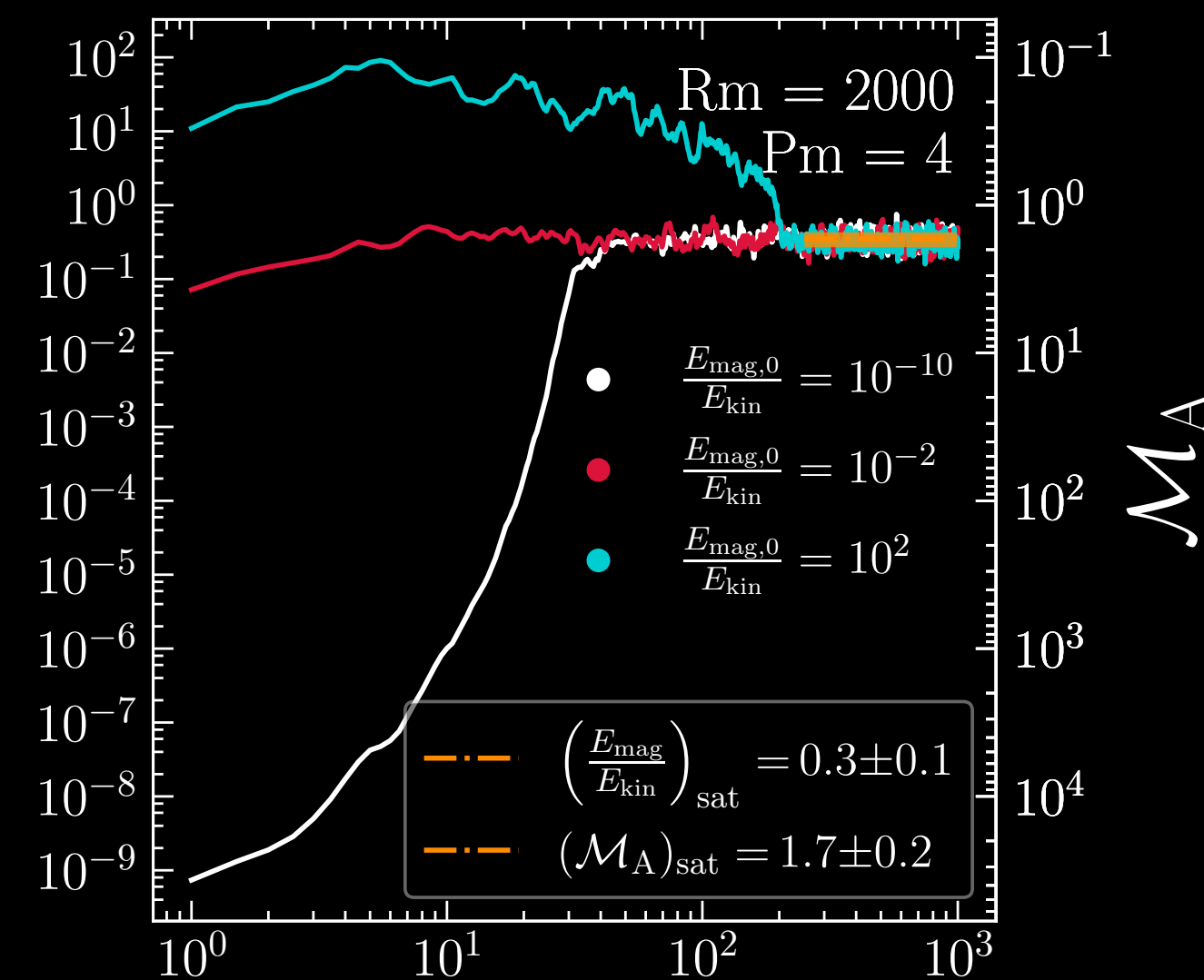
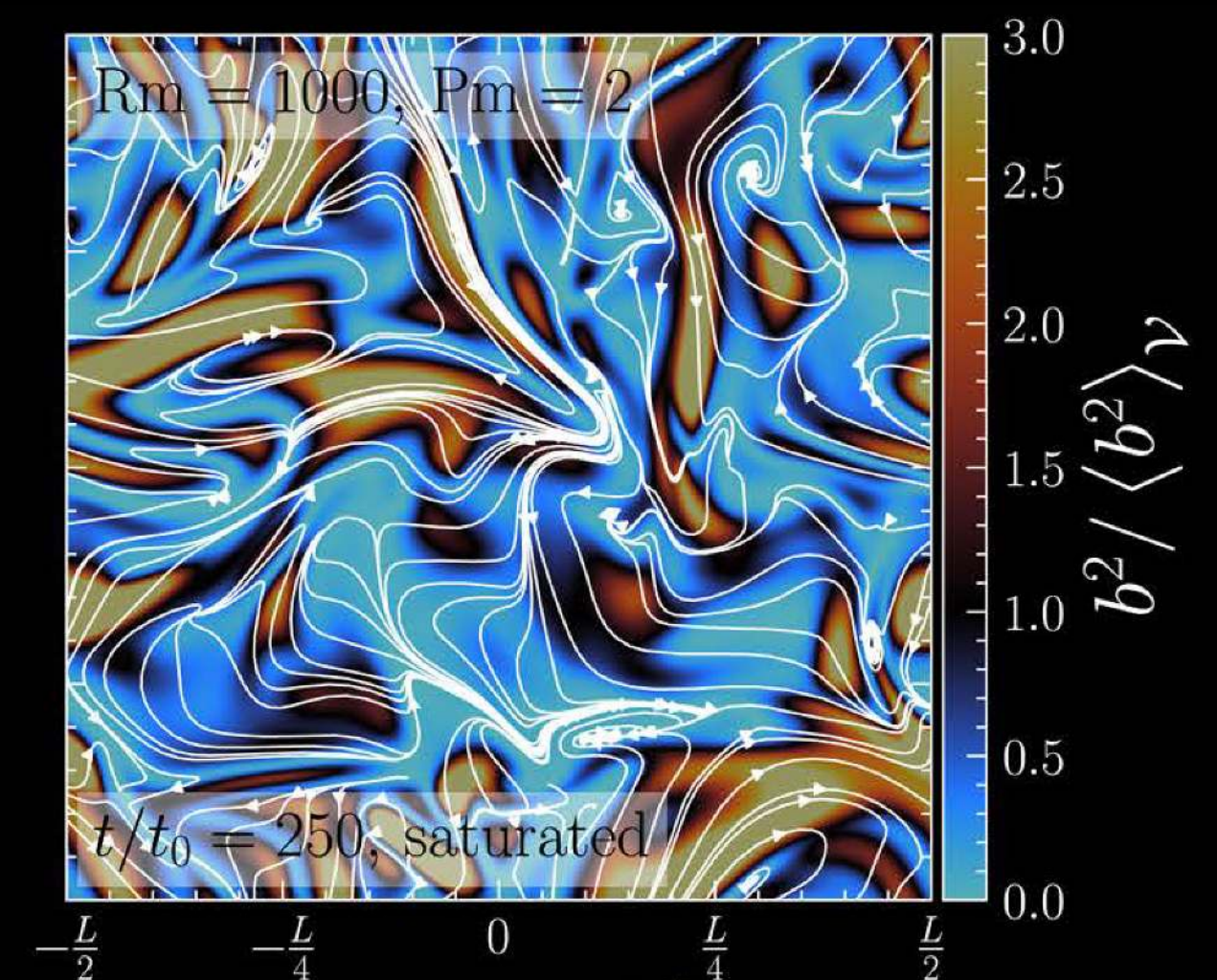
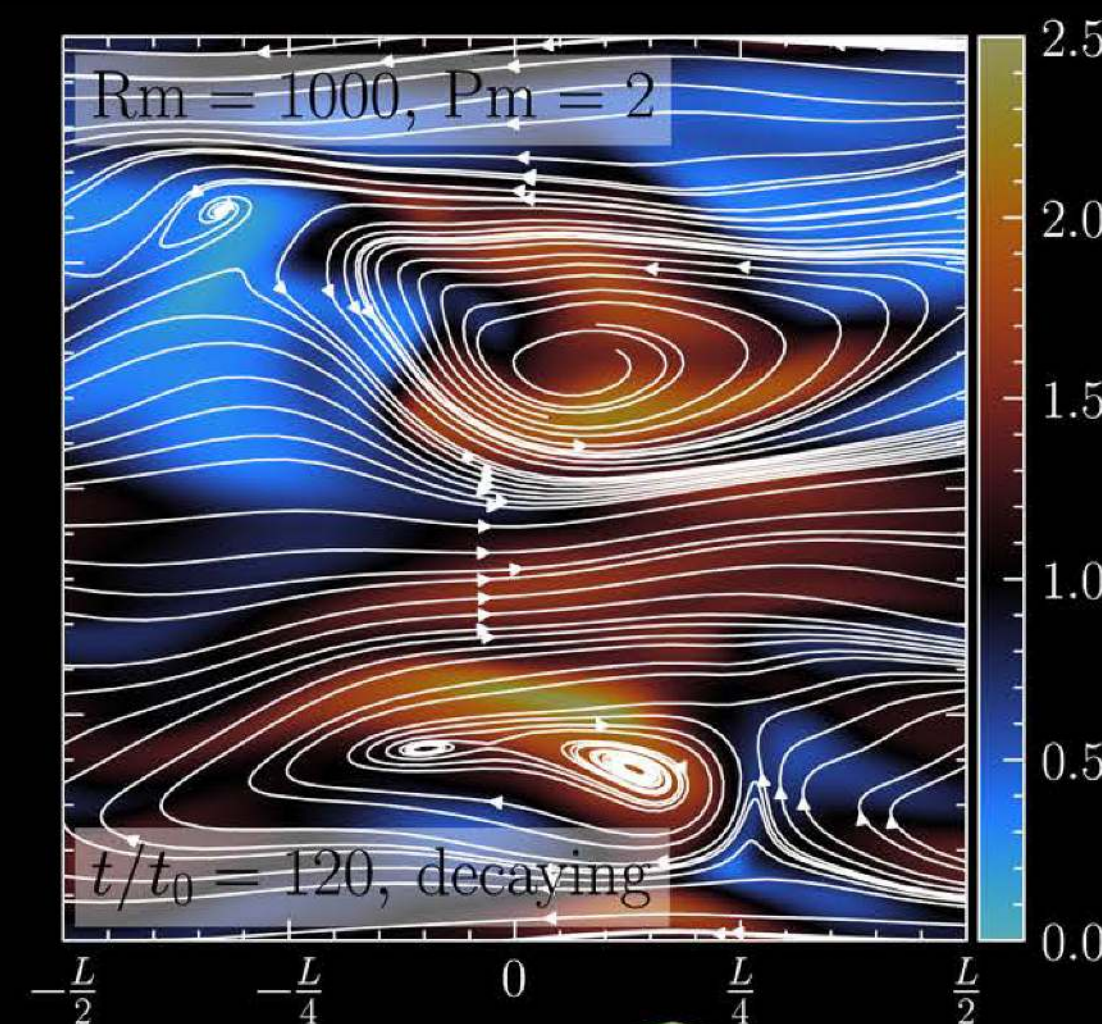
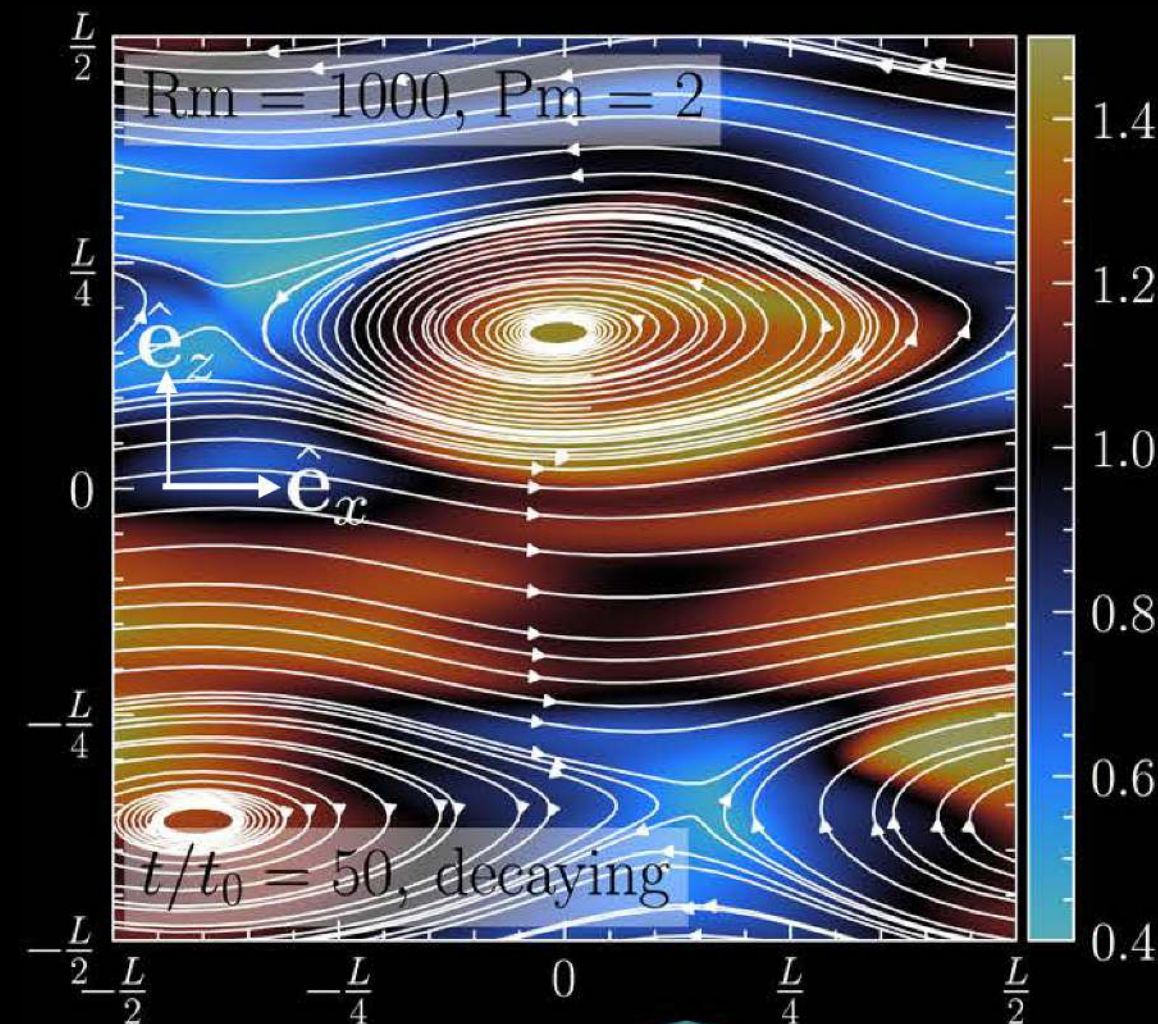
Almost perfect linear Taylor state

$$t = t_{\text{sat}} \quad E_{\text{mag}} \sim E_{\text{kin}}$$

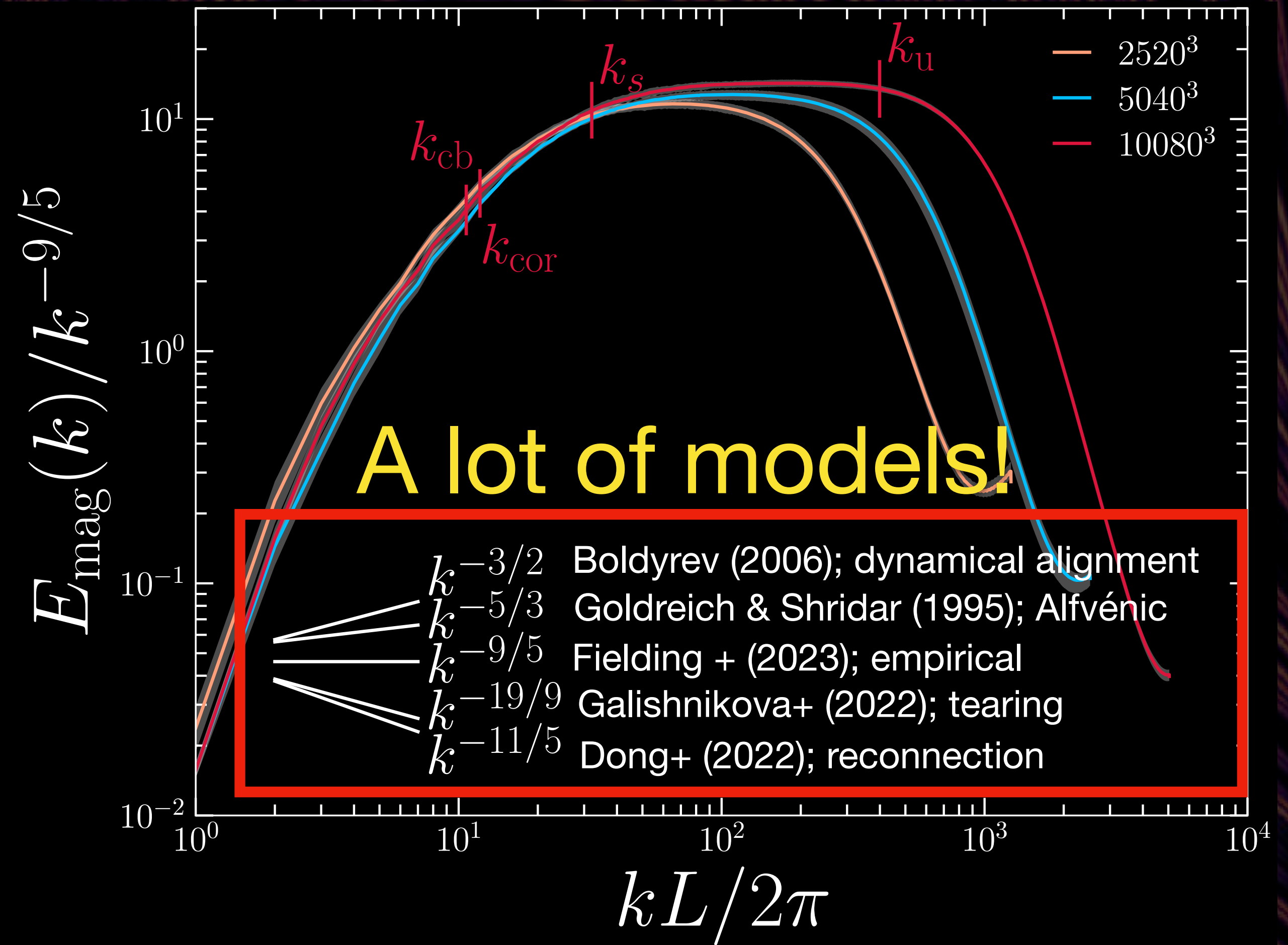
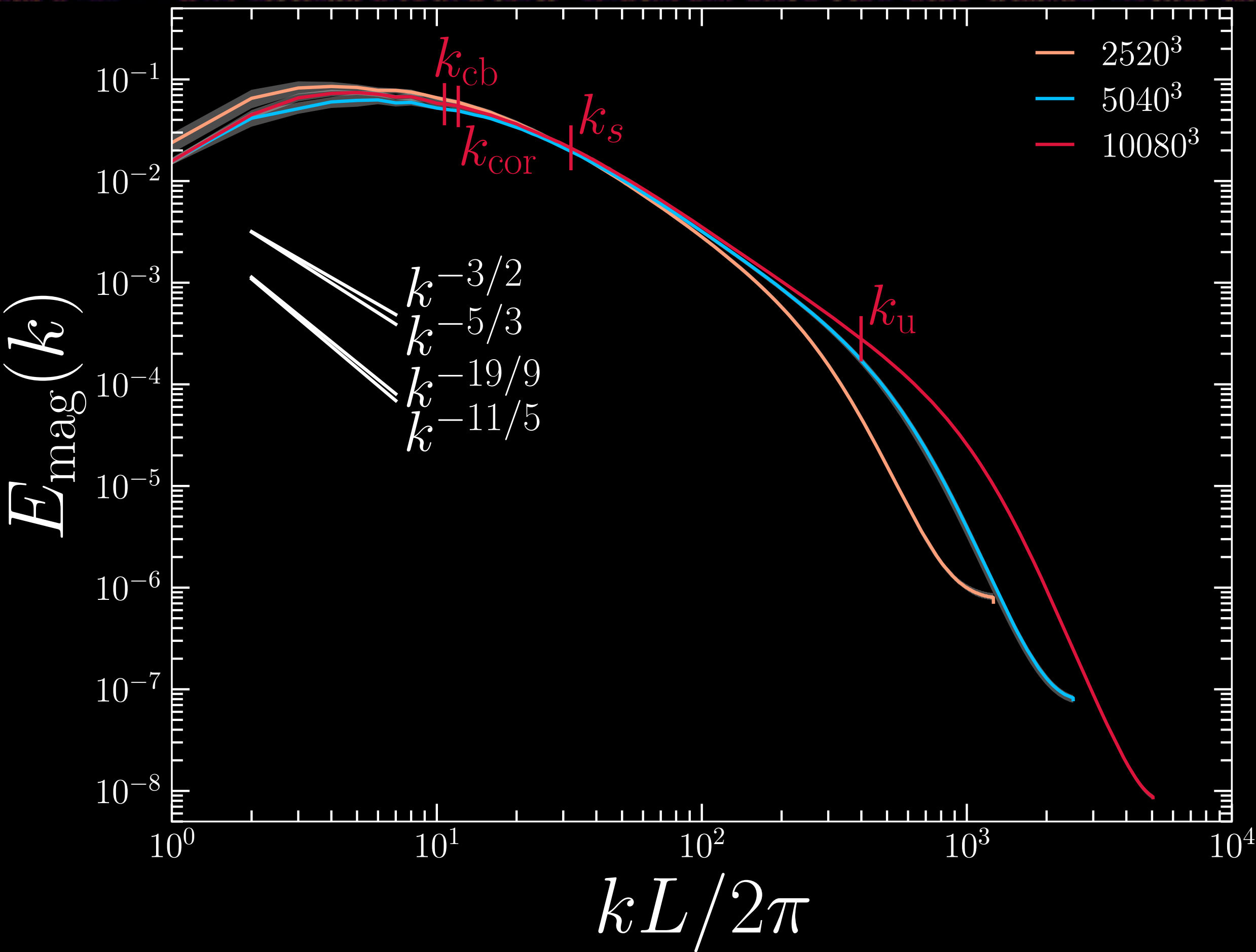
Saturation set by turbulent dynamo

$$\alpha_0 = \frac{\mathbf{j} \cdot \mathbf{b}}{b^2}$$

$$\mathbf{b} = \alpha_0 \mathbf{j}$$



The magnetic energy cascade



The magnetic energy cascade

For $Rm \gtrsim 10^5$ we expect current sheet instabilities on small scales to modify the cascade $E_{\text{mag}} \sim k^{-11/5}$

Biskamp (1987); Lapenta (2008); Bhattacharjee+(2009); Boldyrev & Loureiro (2013); Huang & Bhattacharjee (2013); Dong+(2022)

tearing instabilities growing inside a current sheet

folded magnetic fields from
dynamo

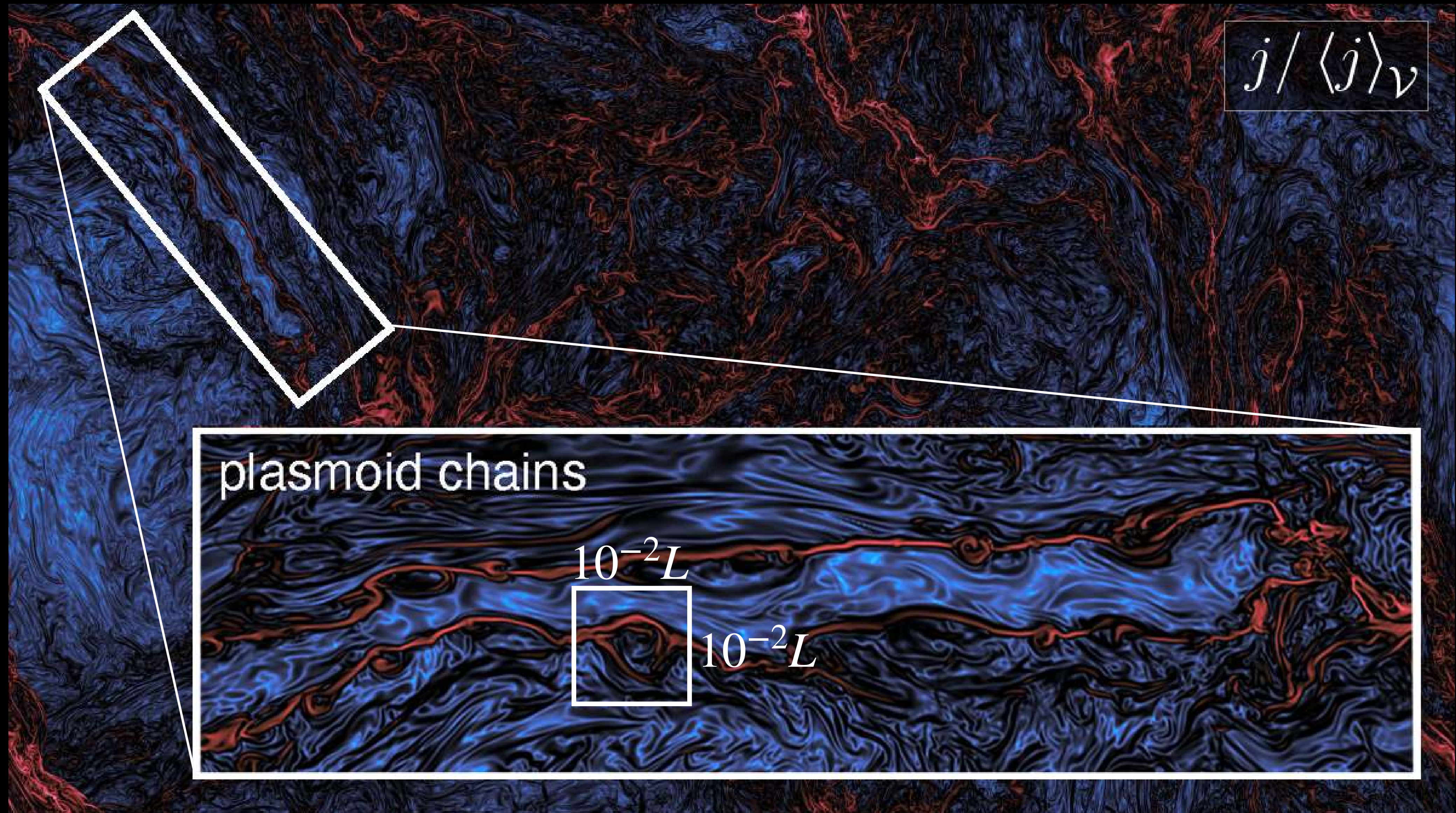
$$l_{\perp} \sim l_{\eta}$$

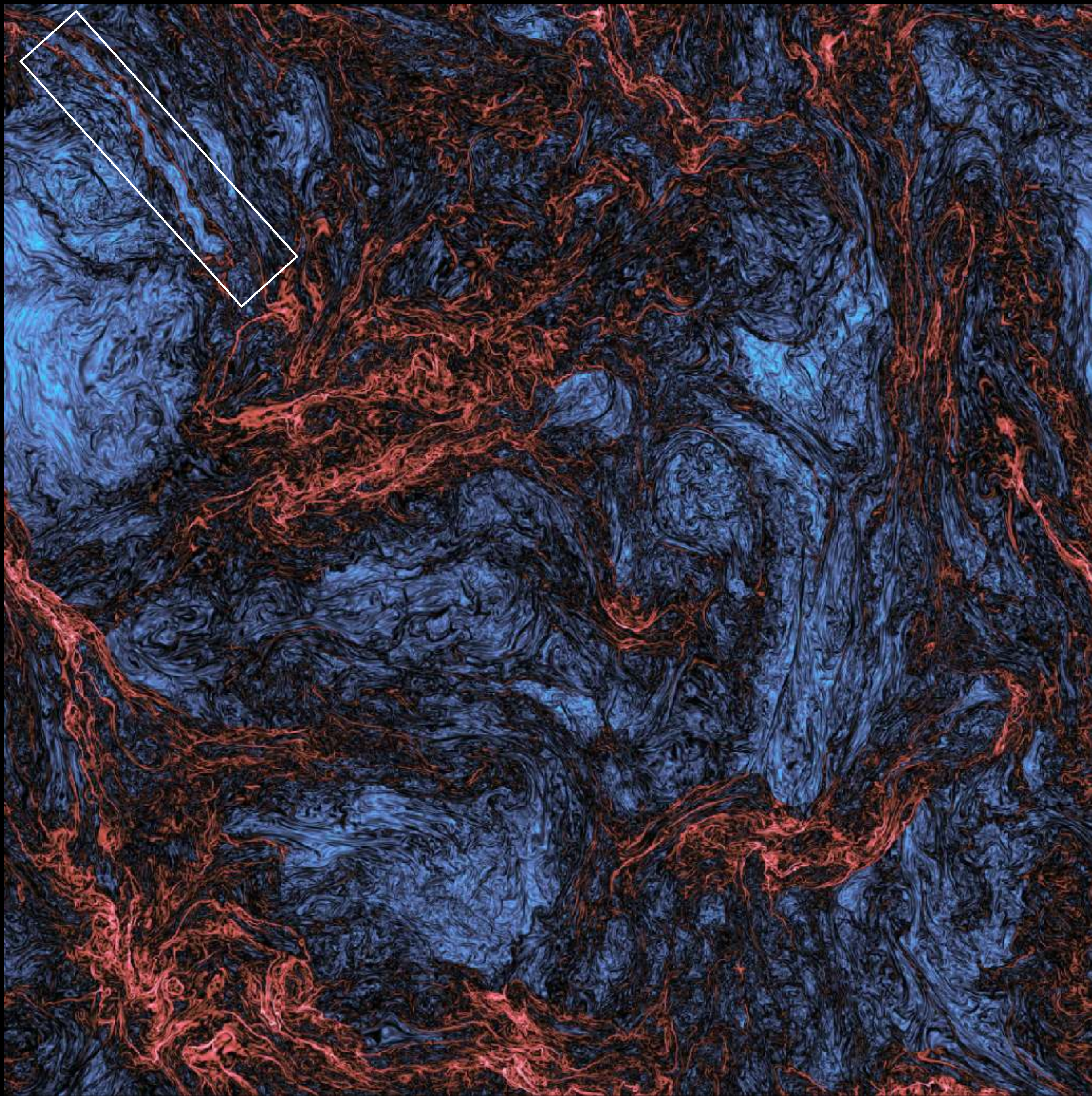
fields reversals

$$l_{\parallel} \sim l_{\nu}$$

Galishnikova+ (2022) PRX

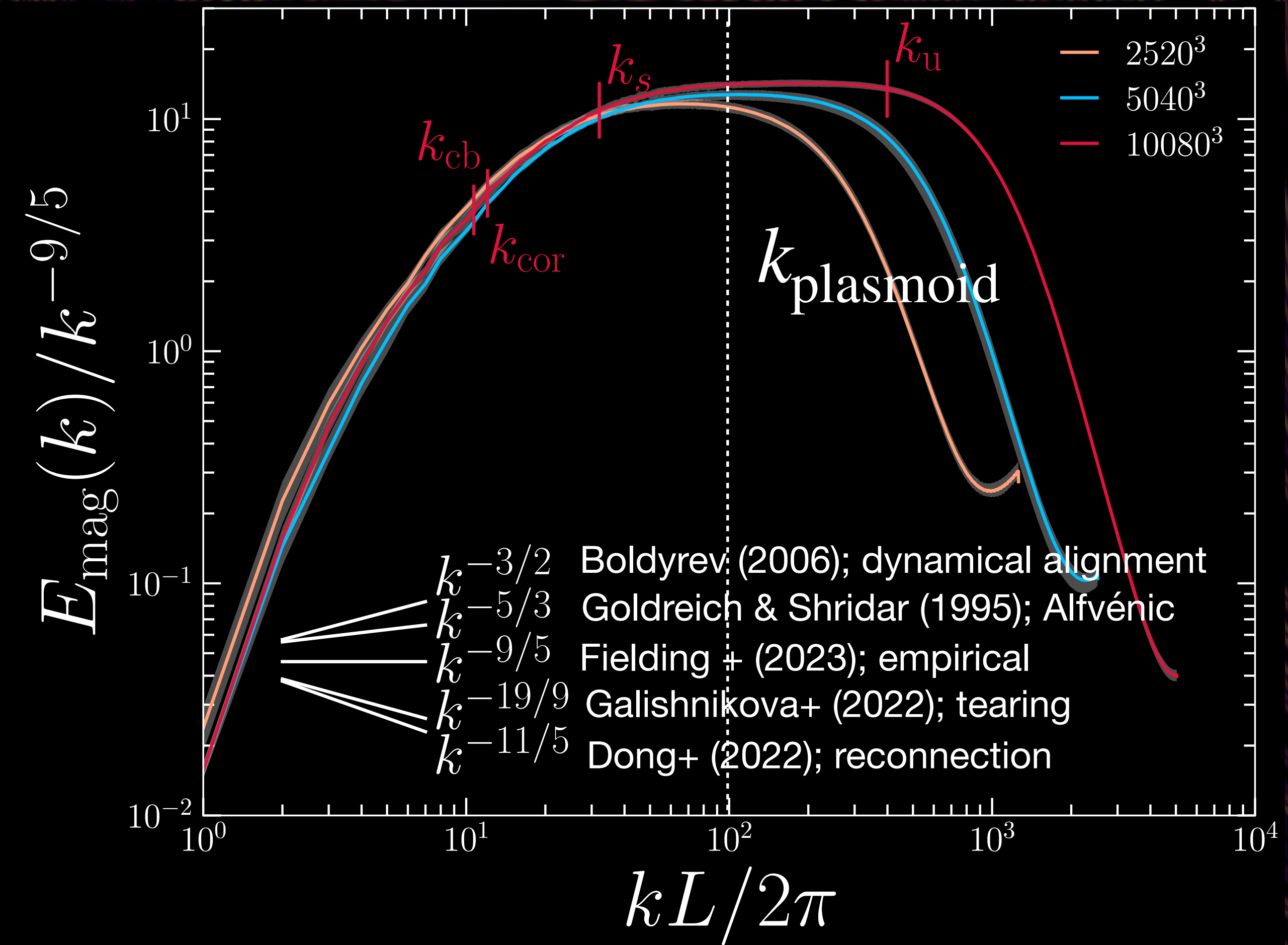
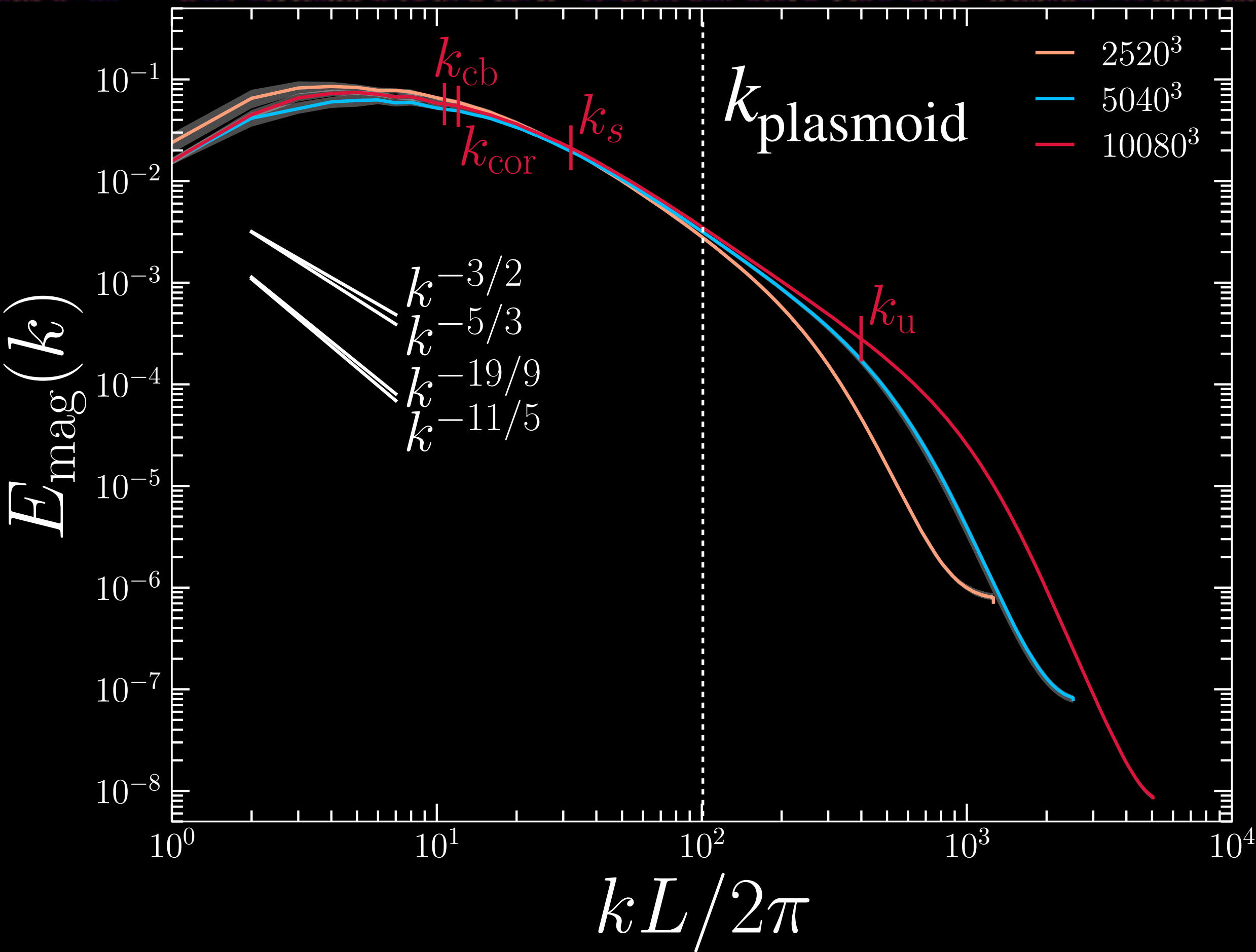
The magnetic energy cascade



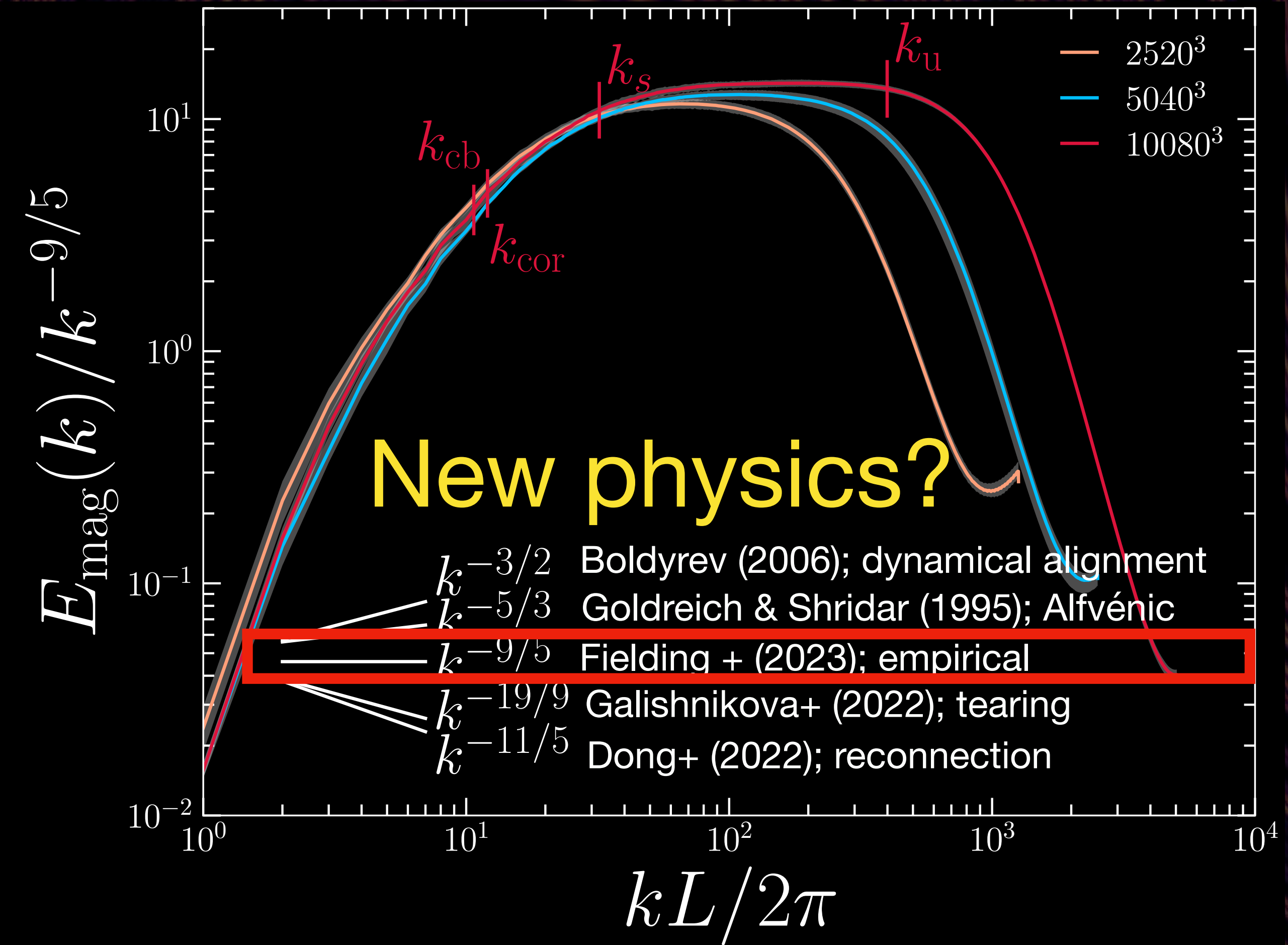
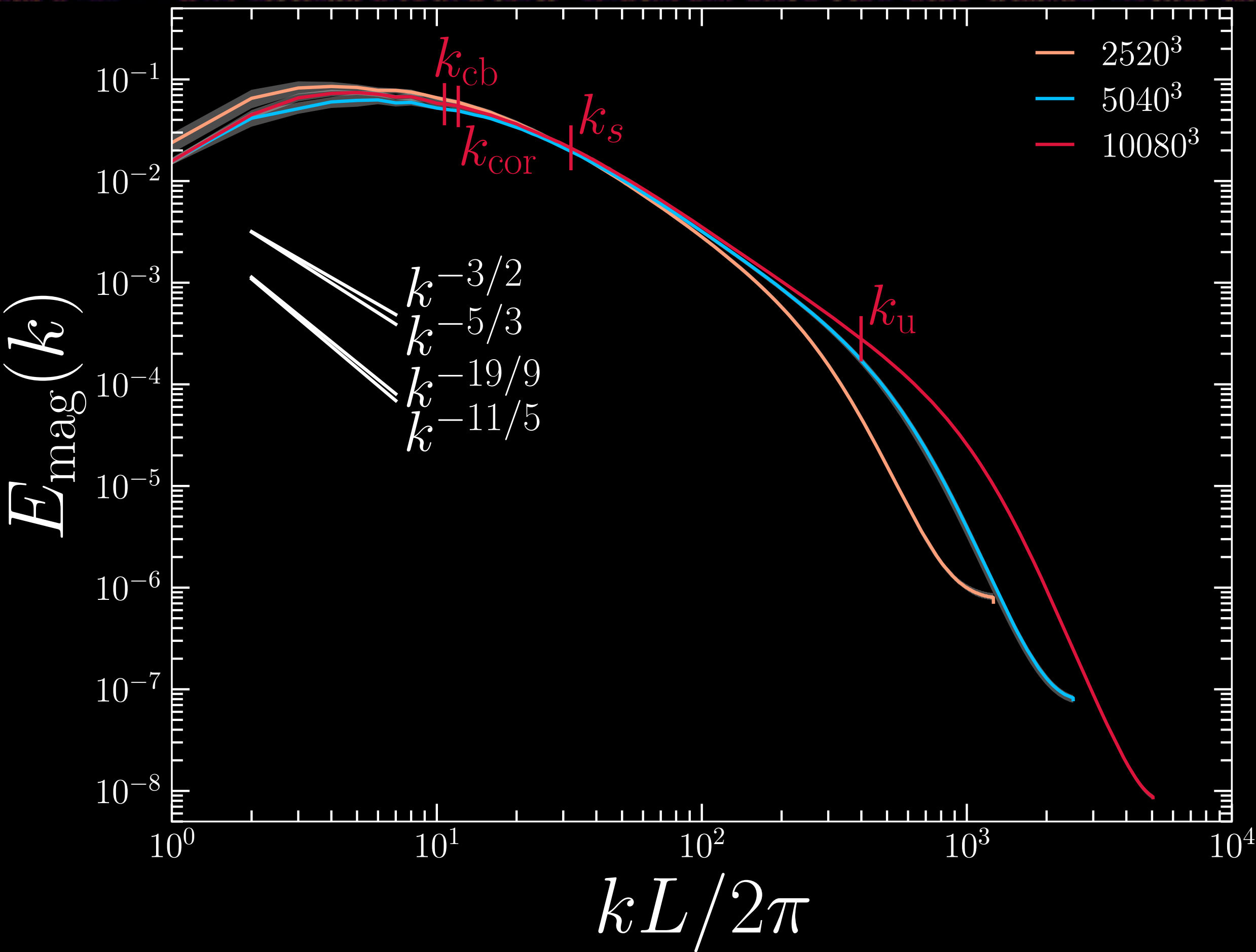


Occupy a very small
volume filling-factor
in this regime

The magnetic energy cascade



The magnetic energy cascade



Summary

Strong, supersonic, hydrodynamical cascade:

$$k_0 < k < k_{\text{eq}}, \quad E_{\text{kin}} \sim k^{-2}$$

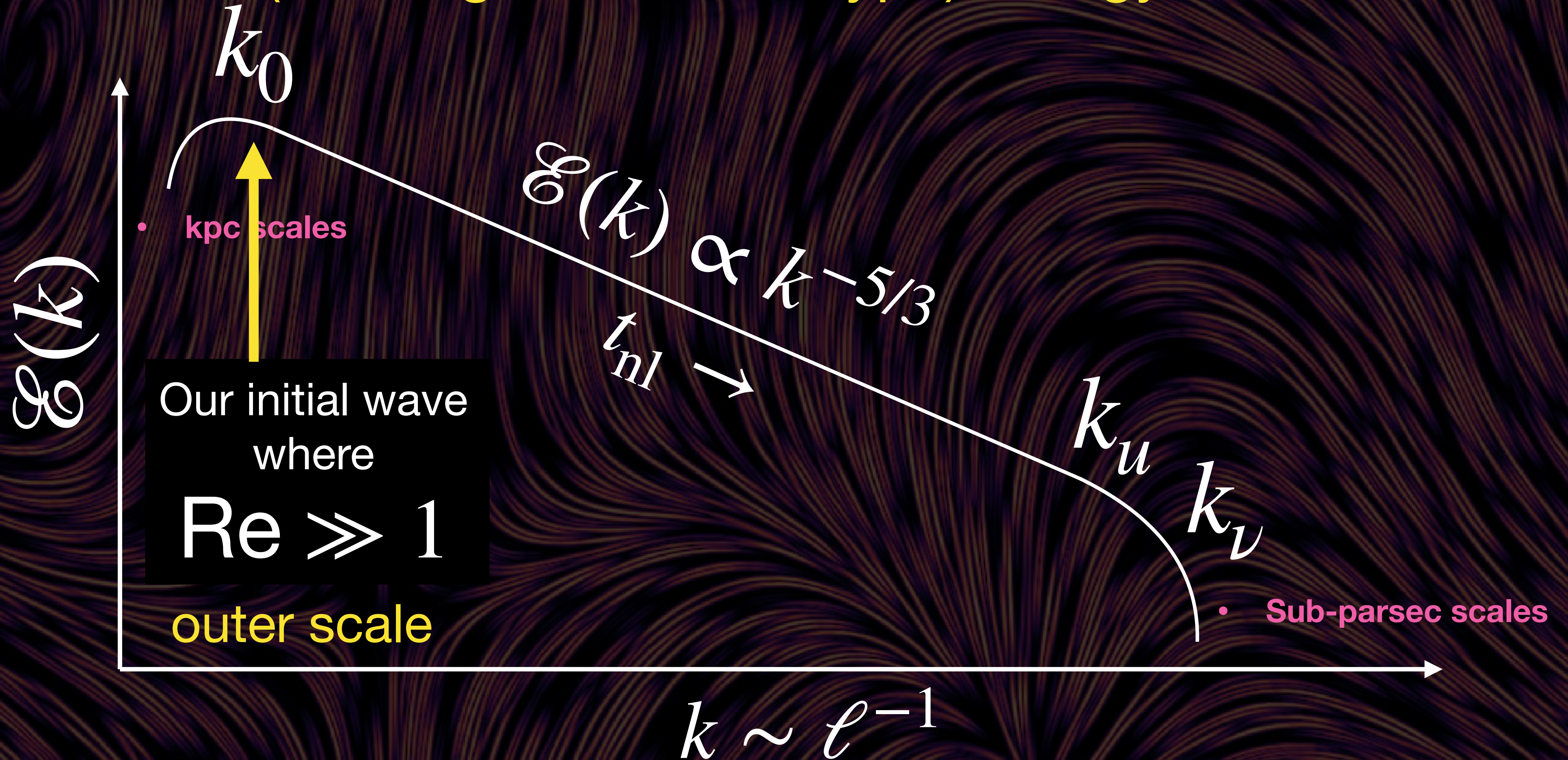
Subsonic, weakened nonlinear magnetohydrodynamical cascade that becomes magnetically dominated, relaxed on small scales:

$$k_{\text{eq}} < k < k_{\text{u}}, \quad E_{\text{kin}} \sim k^{-3/2}$$

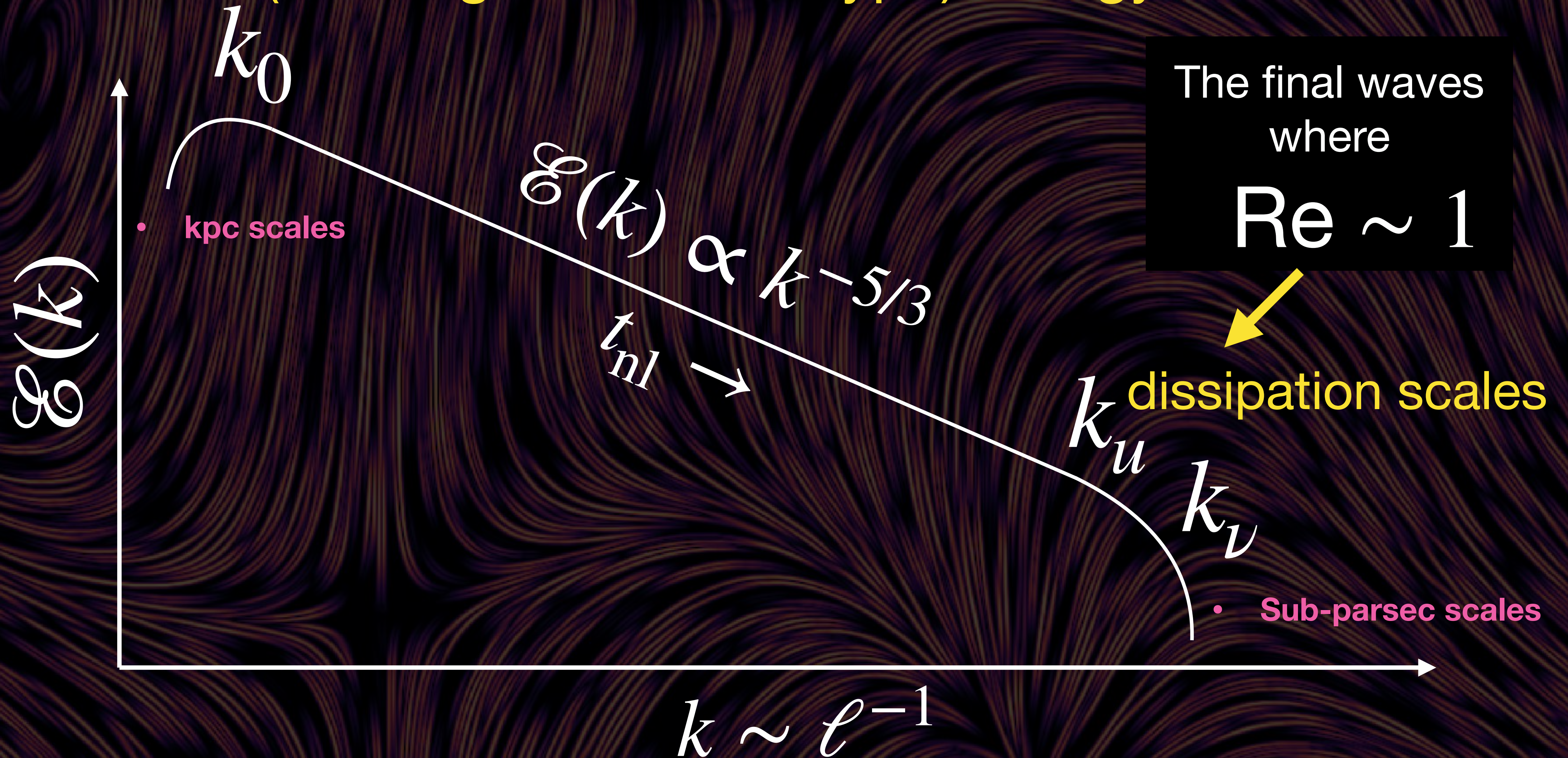
A magnetic cascade develops at very high-Rm:

$$k_{\text{s}} < k < k_{\text{u}}, \quad E_{\text{mag}} \sim k^{-9/5}$$

The (Kolmogorov, 1941 -type) energy cascade

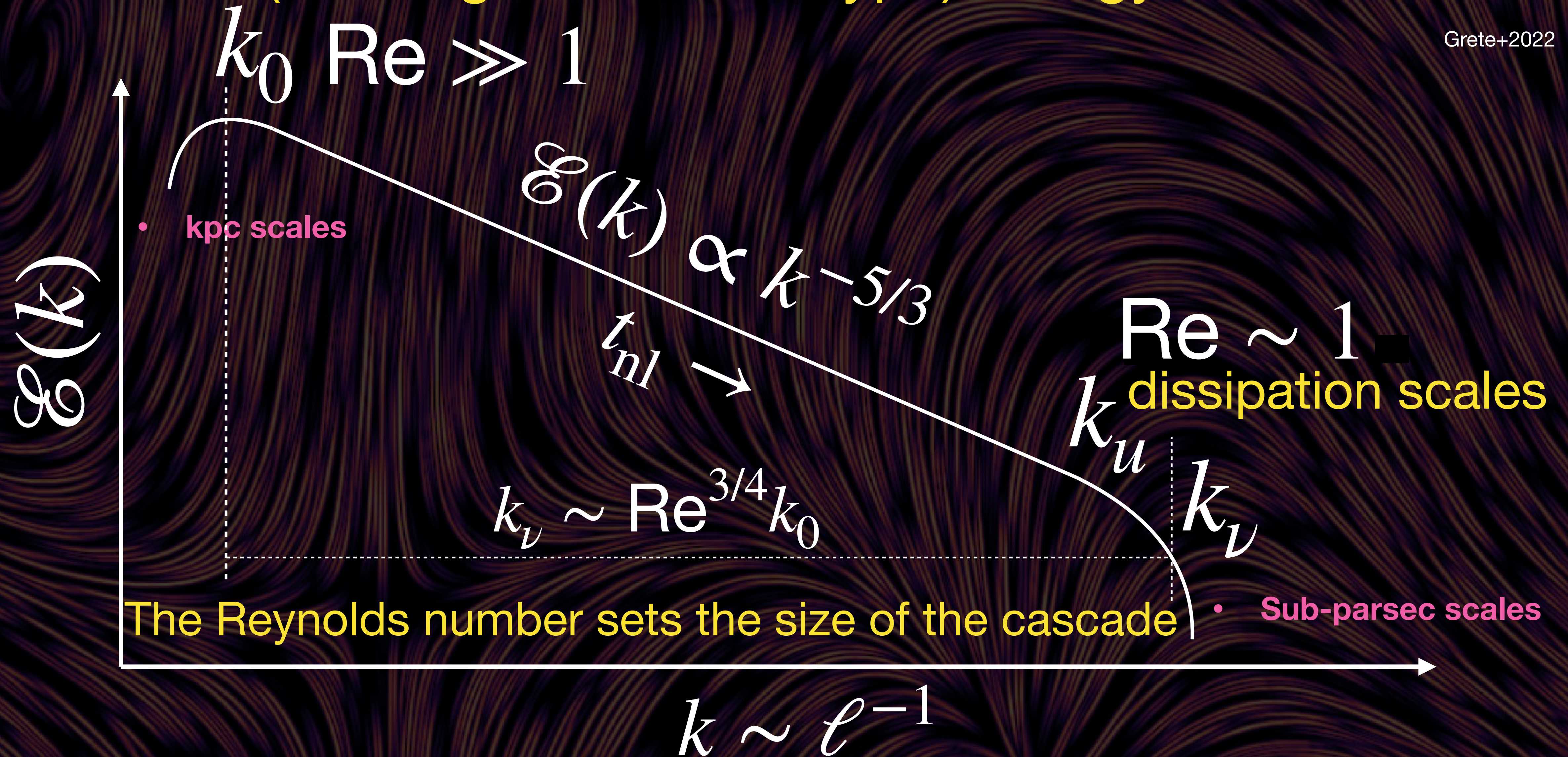


The (Kolmogorov, 1941 -type) energy cascade

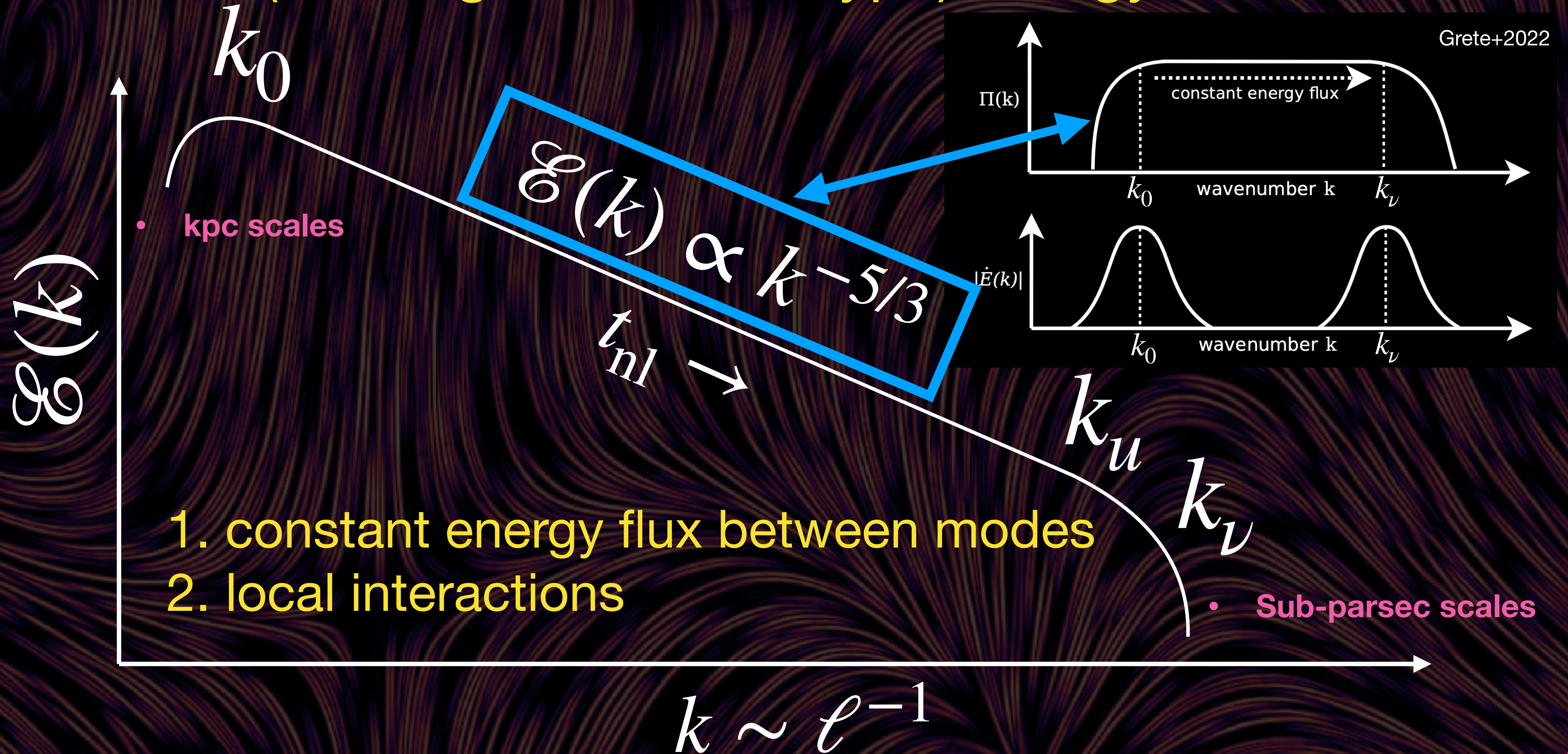


The (Kolmogorov, 1941 -type) energy cascade

Grete+2022



The (Kolmogorov, 1941 -type) energy cascade



$$\mathcal{M}_A > 1$$

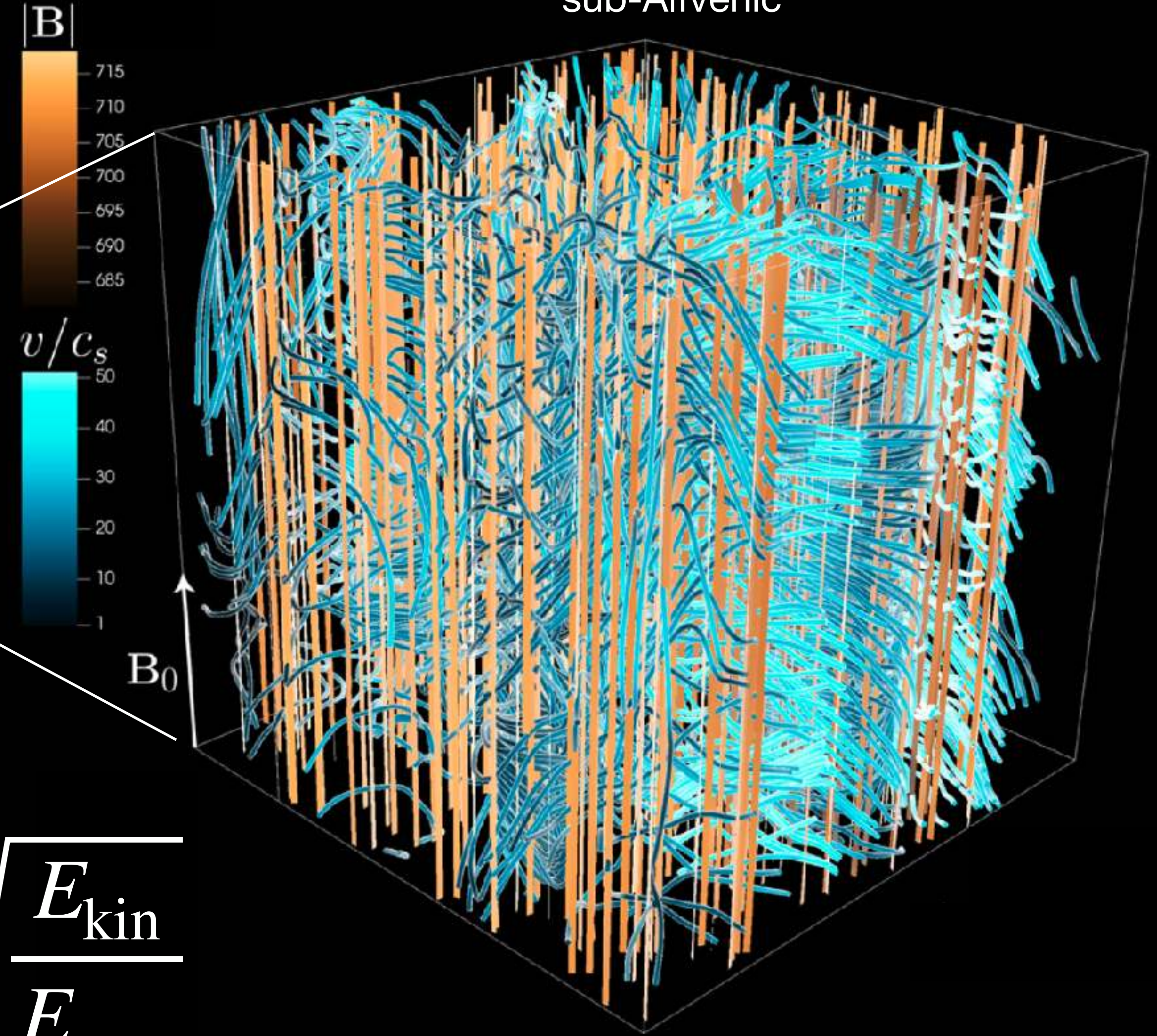
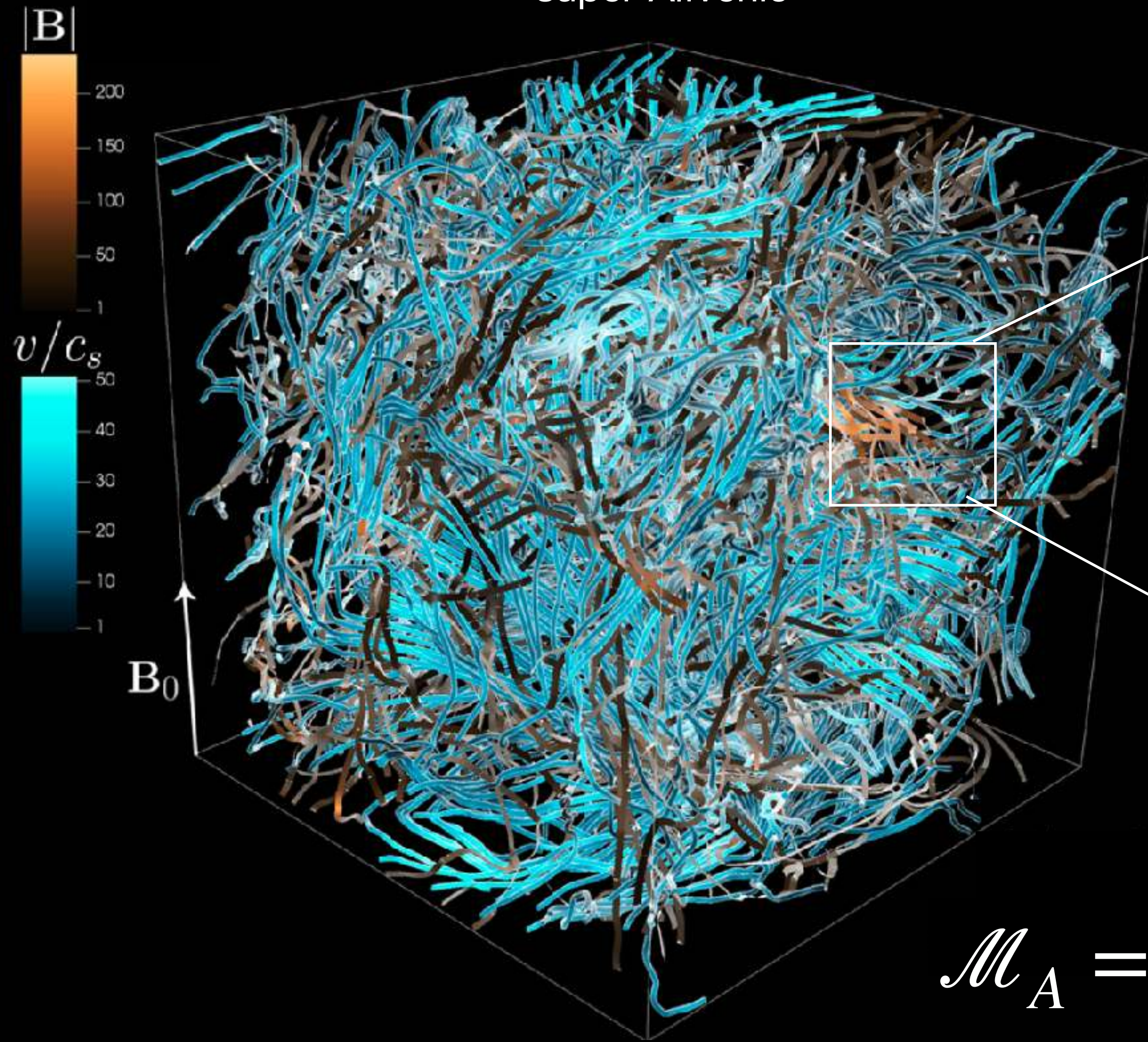
super-Alfvénic

$$\kappa \propto \mathcal{M}_A^{1/2}$$

Schekochihin+2004

$$\mathcal{M}_A < 1$$

sub-Alfvénic

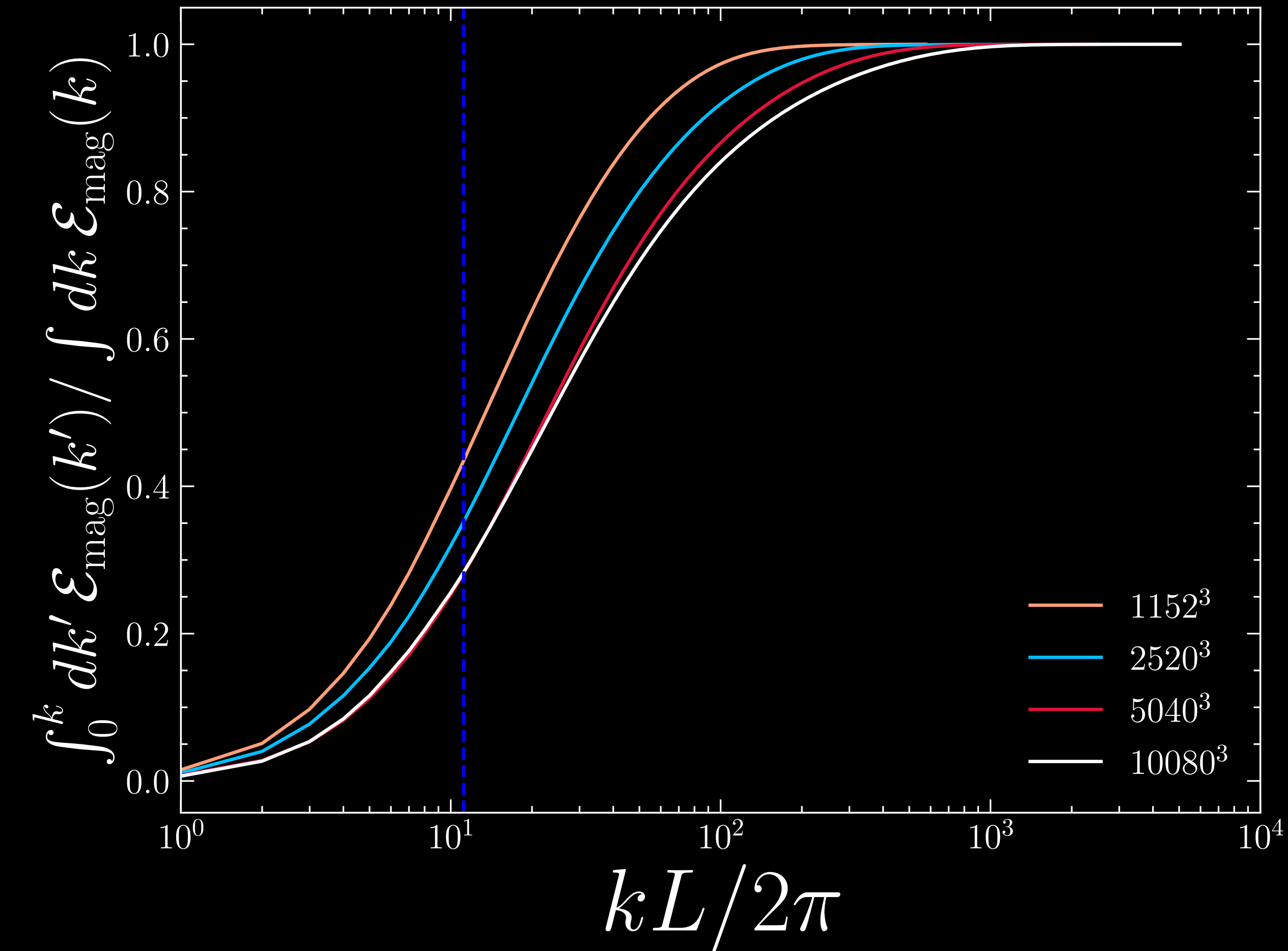


$$\mathcal{M}_A = \sqrt{\frac{E_{\text{kin}}}{E_{\text{mag}}}}$$

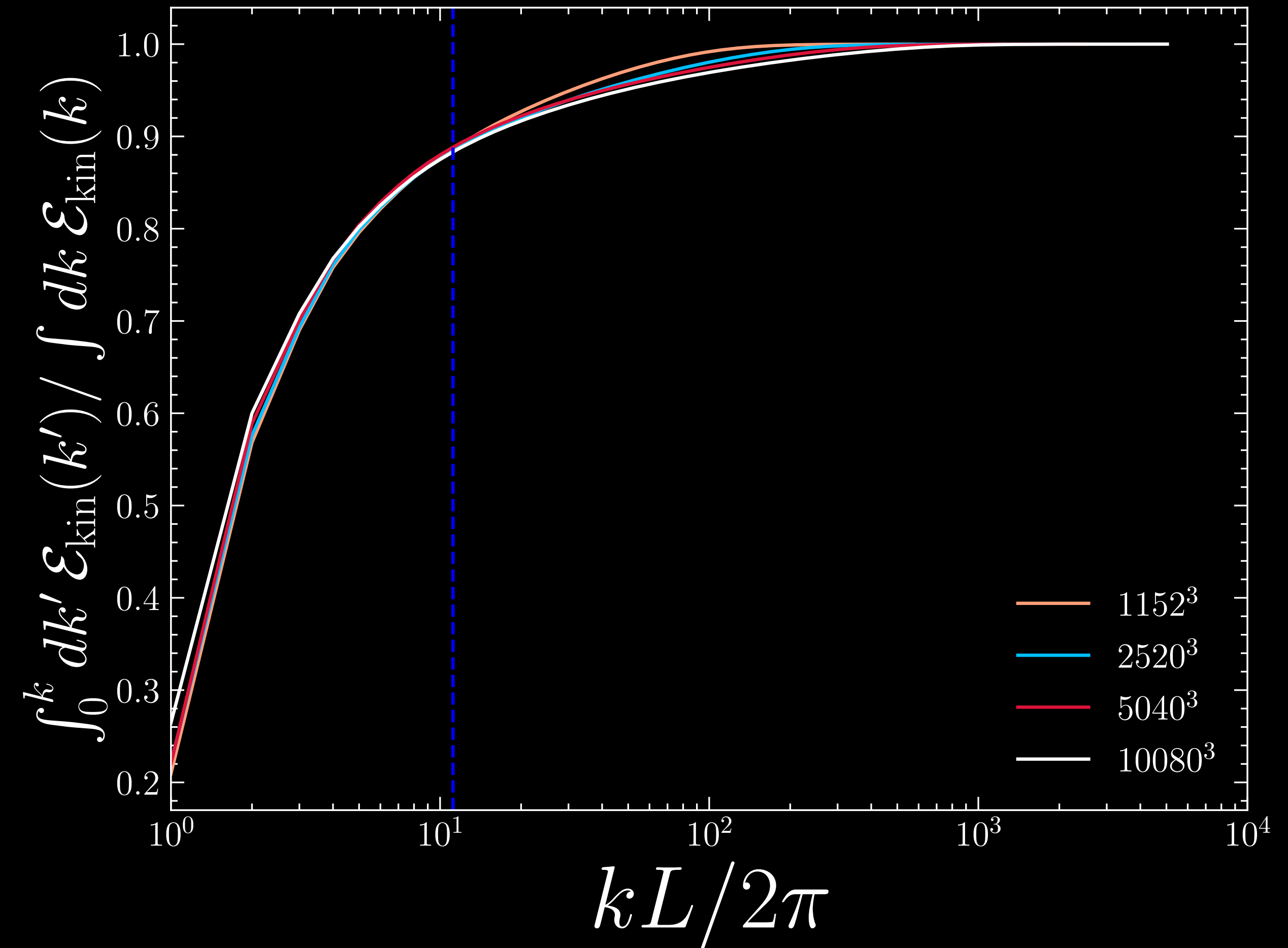
classically strong turbulence

classically weak turbulence

Cumulative magnetic energy

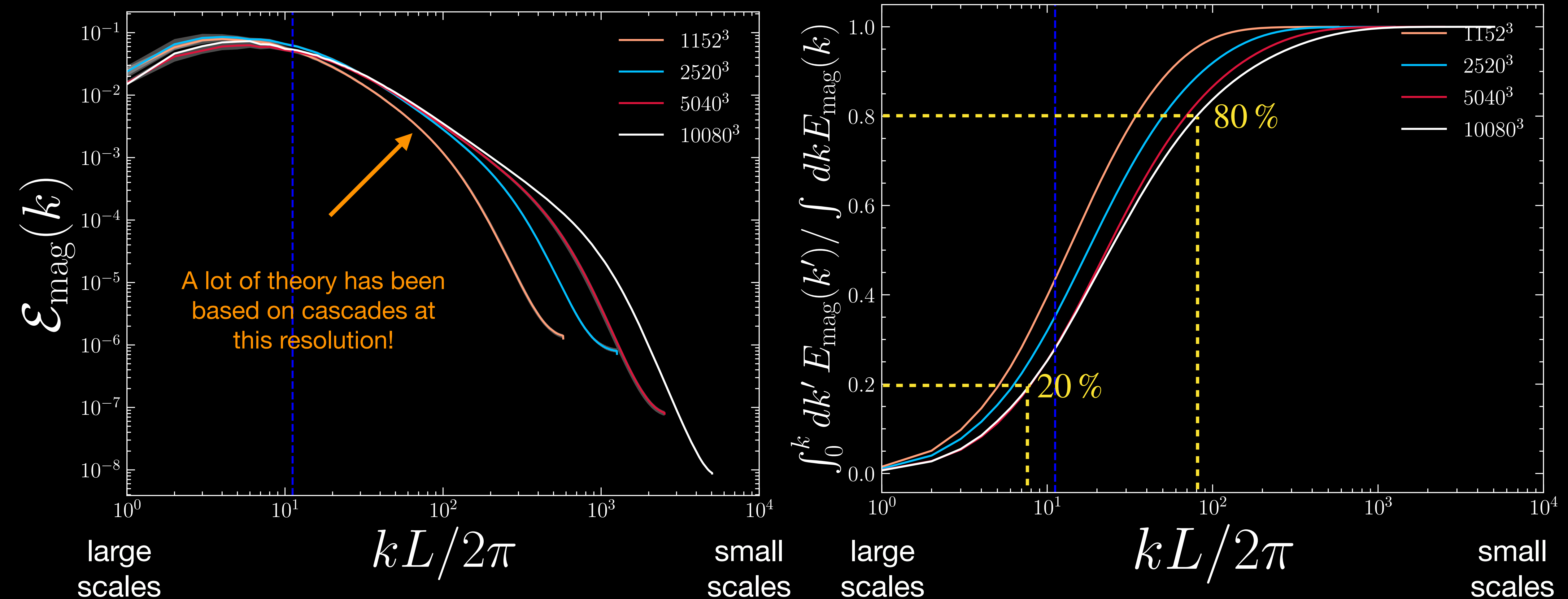


Cumulative kinetic energy



(preliminary results)

Magnetic energy cascade (preliminary results)

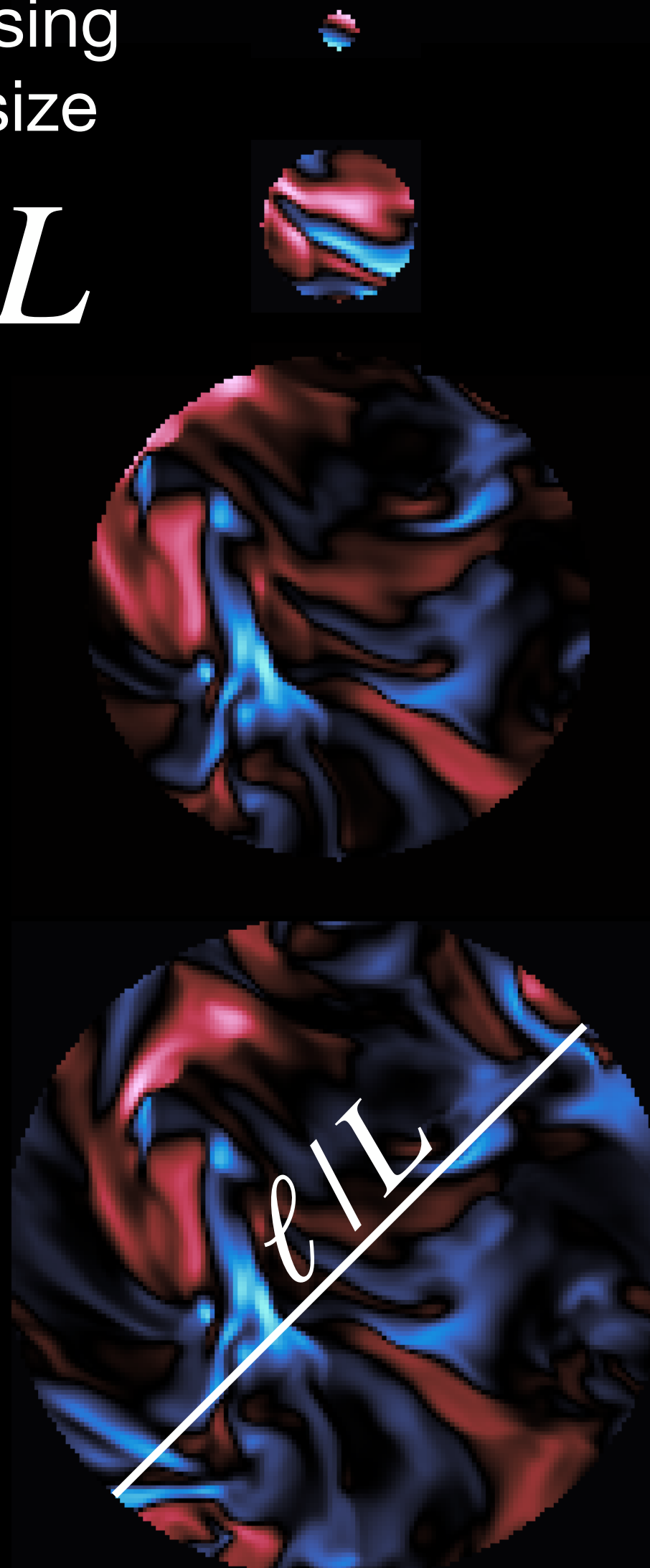


Magnetic field is intrinsically a small-scale field! Need a lot of resolution to resolve it properly (energetically)!

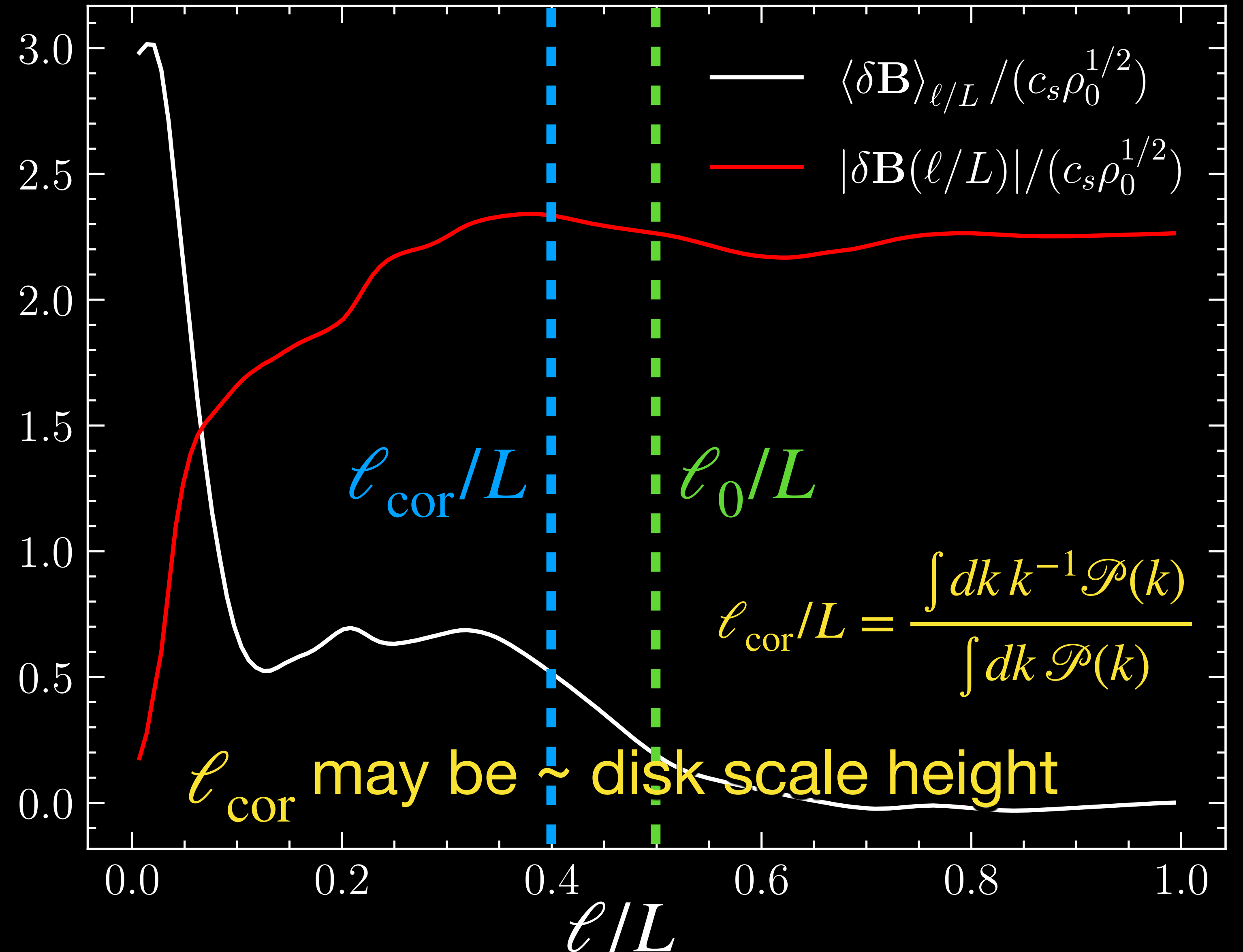
One such something: measuring turbulence below ℓ_{cor}

Increasing
filter size

ℓ/L



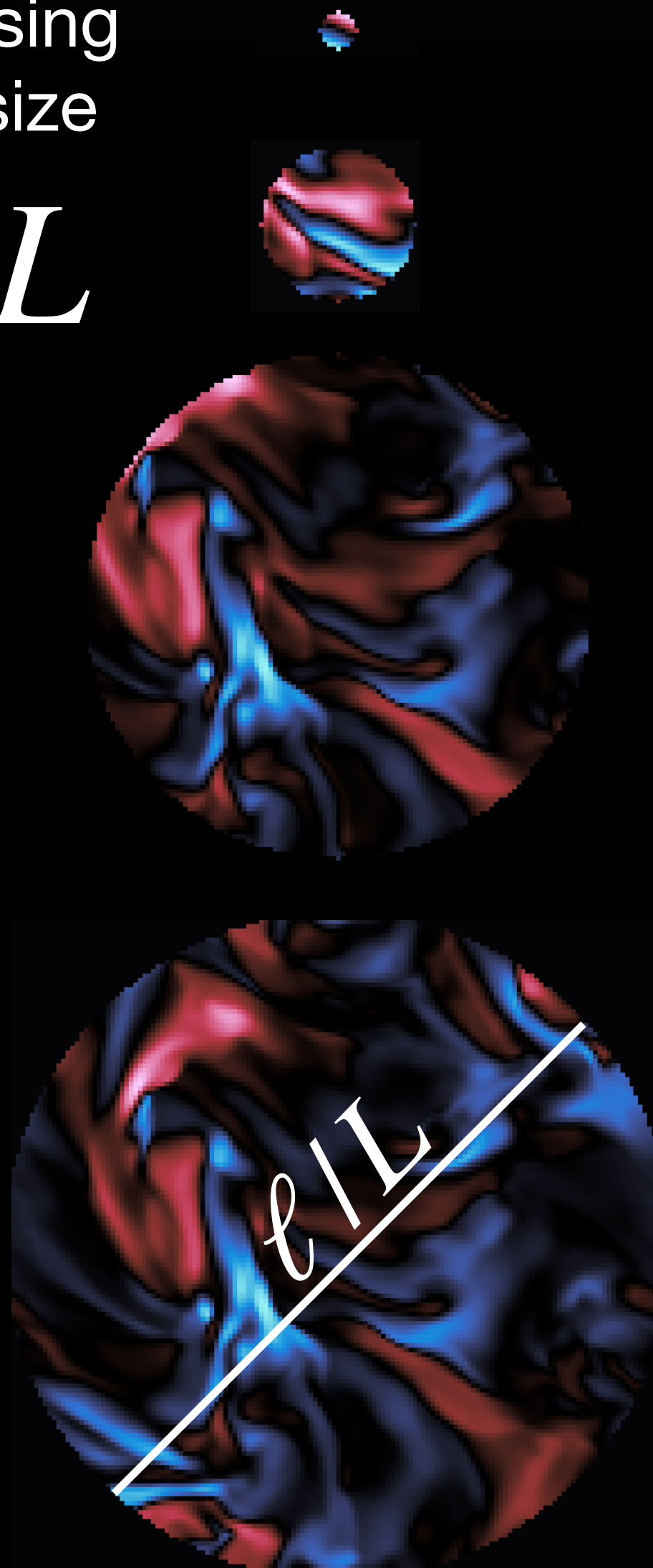
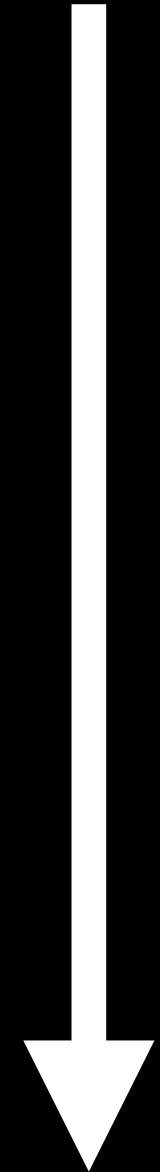
$B(\ell/L)/(c_s \rho_0^{1/2})$



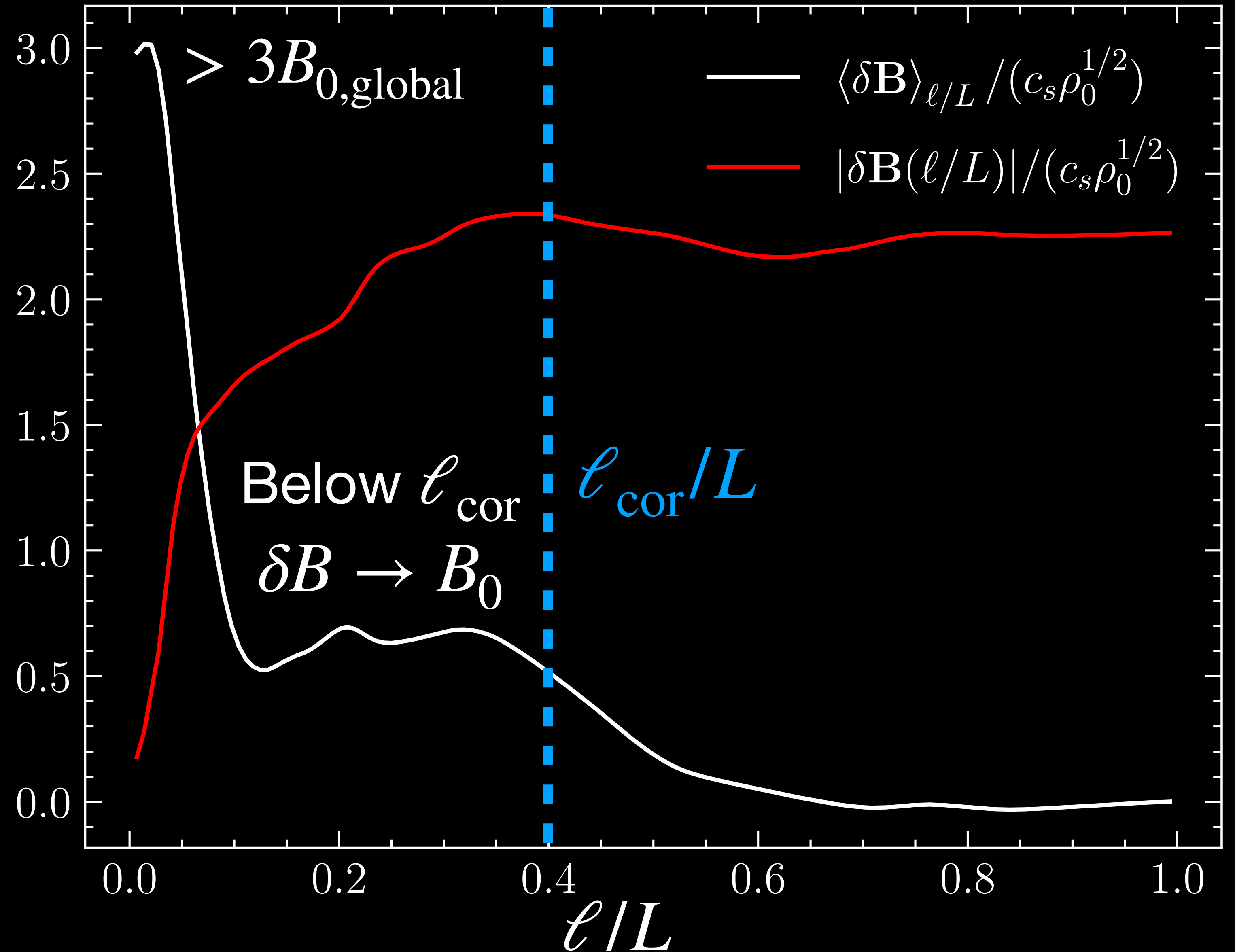
One such something: measuring turbulence below ℓ_{cor}

Increasing
filter size

ℓ/L



$B(\ell/L)/(c_s \rho_0^{1/2})$



Peta-scale simulations

World's largest compressible MHD turbulence simulation

Beattie, Federrath, Klessen, Mocz & Cielo

1. What is the exact position of the magnetosonic scale?
2. What is the scaling of the energy cascades in compressible MHD turbulence?

70million core-hour project on superMUC-NG

ILES of comp. MHD turbulence

Turbulence: $\mathcal{M} \approx 4$, $\ell_0 = L/2$, $\text{Re} \sim 10^6$

Magnetic fields: $B = b_{\text{turb}}$, $\mathcal{M}_A \approx 2$, $\text{Rm} \sim 10^6$

Including $\approx 10^{11}$ Lagrangian tracer particles

Three experiments for convergence tests:

- **LOW-RES:** $2,520^3$ (0.3Mcore-h, 8,640cores)
- **MID-RES:** $5,040^3$ (8.0Mcore-h, 69,120cores)
- **HIGH-RES:** $10,080^3$ (62.0Mcore-h, 138,240cores)
 - Resolving 10pc down to $\sim 200\text{au}$ everywhere on the grid
 - Factor of 4 higher grid resolution than IllustrisTNG

3.45PB in data products / 1.34PB of memory at runtime

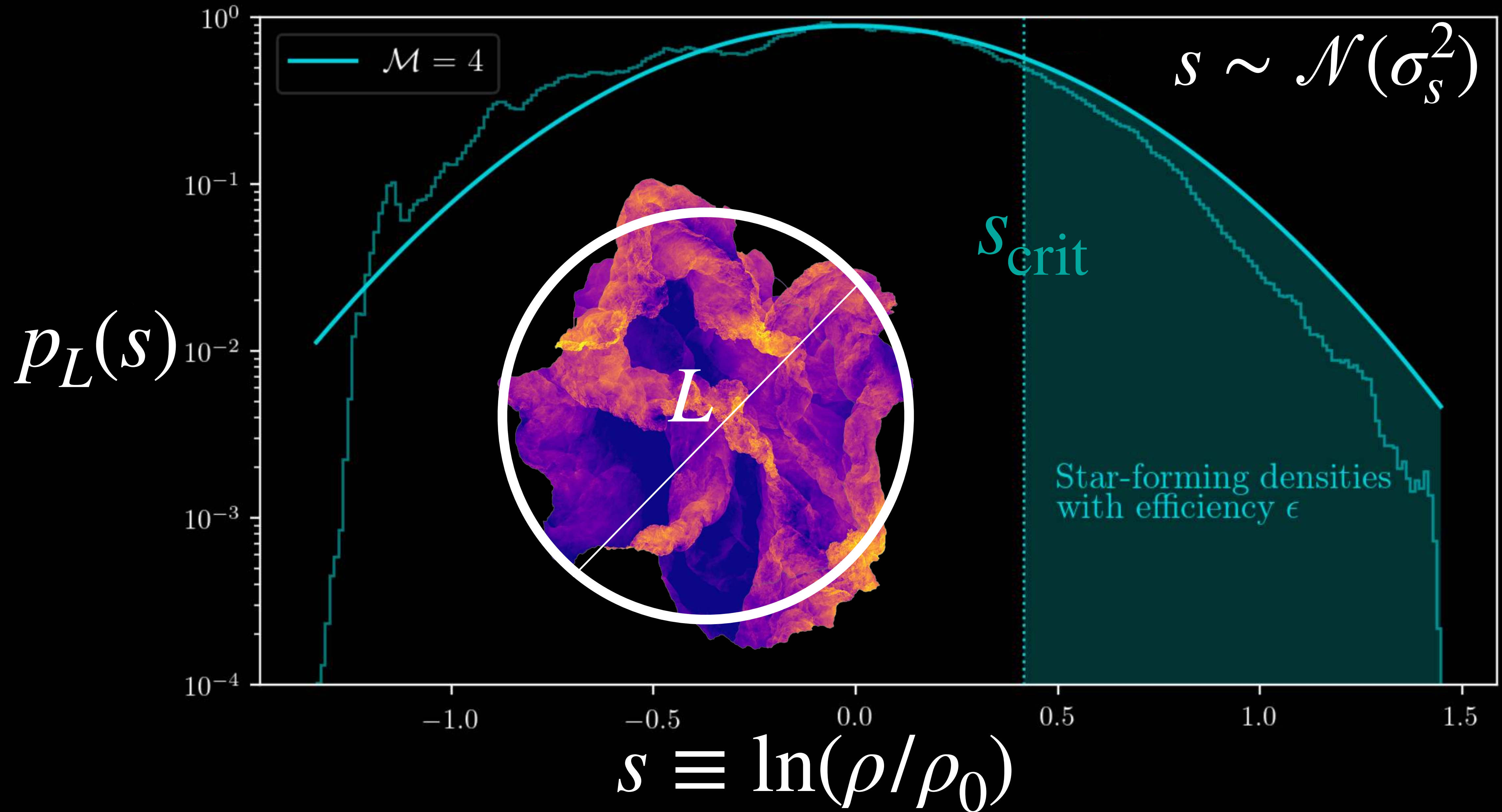
Based on the FLASH code but extremely modified, now only utilises I/O, mesh and particle functionality from the public repository.

Optimisations for running on 100,000s of compute cores:

- Interpolating low-res turbulent fields for initial conditions in high-res experiments
- File splitting for checkpoint and plotfiles to reach theoretical max. file I/O on disk
- Hybrid precision calculations, reducing memory footprint by a factor of 2
- Gas and Alfvén velocity limiters set based on the physics of turbulent fluctuations
- Only store primitive fluid field variables
- Inline calls to equation of state
- Optimise calls to integral flow quantities

Turbulence-regulated star-formation based on $s \equiv \ln(\rho/\rho_0)$ statistics

Krumholz & McKee (2005); Federrath & Klessen (2012); Mocz & Burkart (2018), Burkart & Mocz (2018)



Sonic scale

Federrath, Klessen, Ipachio & Beattie (2021)

Nature Astronomy

Federrath & Klessen (2012)

L

$$\sigma_v(\ell/L) > c_s$$

P_{therm}

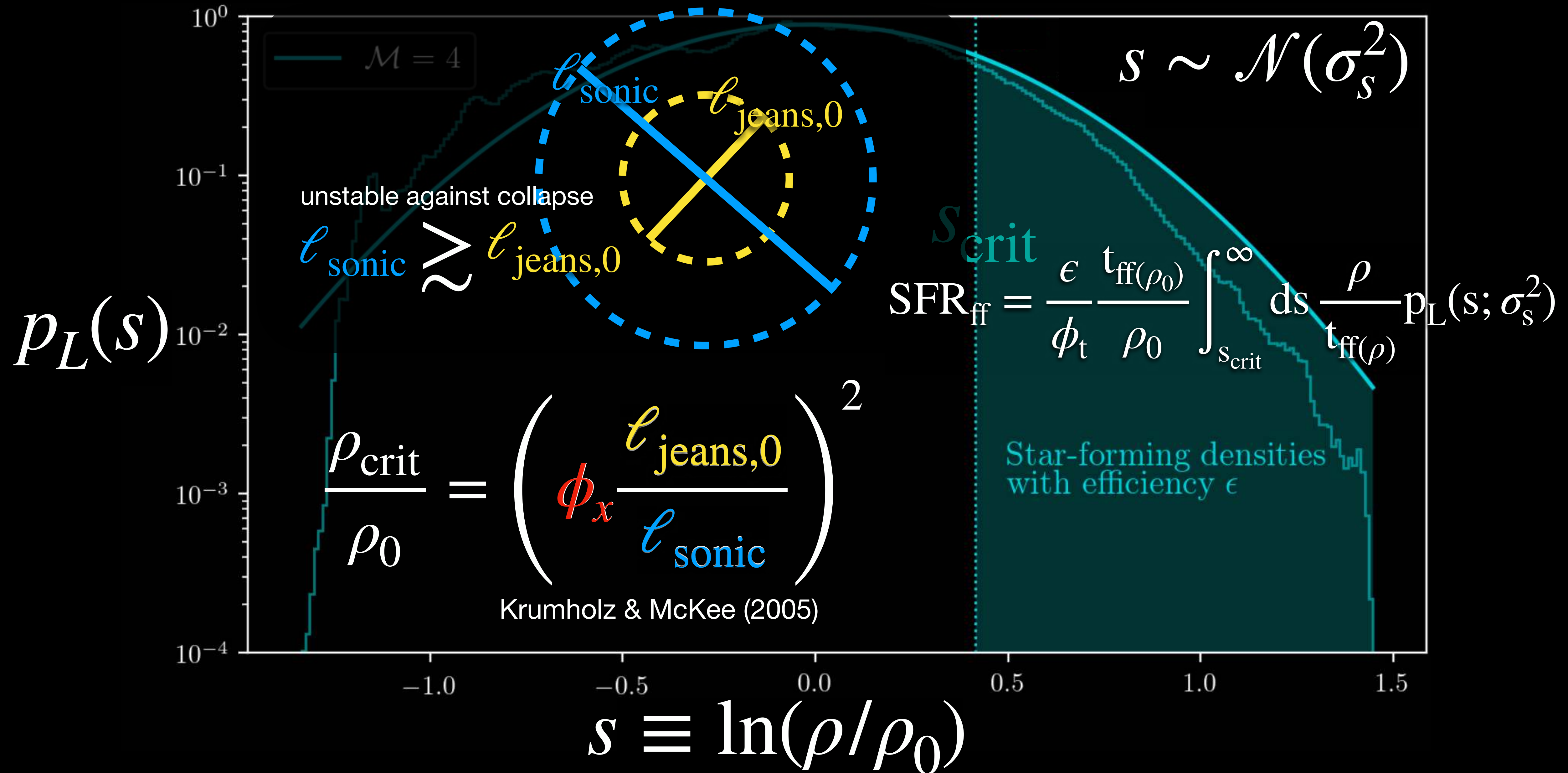


$$\sigma_v(\ell_s/L) = c_s$$
$$\ell_s \propto L \mathcal{M}^{-2}$$

$$\mathcal{M} = \frac{\langle \delta v^2 \rangle_{\mathcal{V}}^{1/2}}{c_s}$$

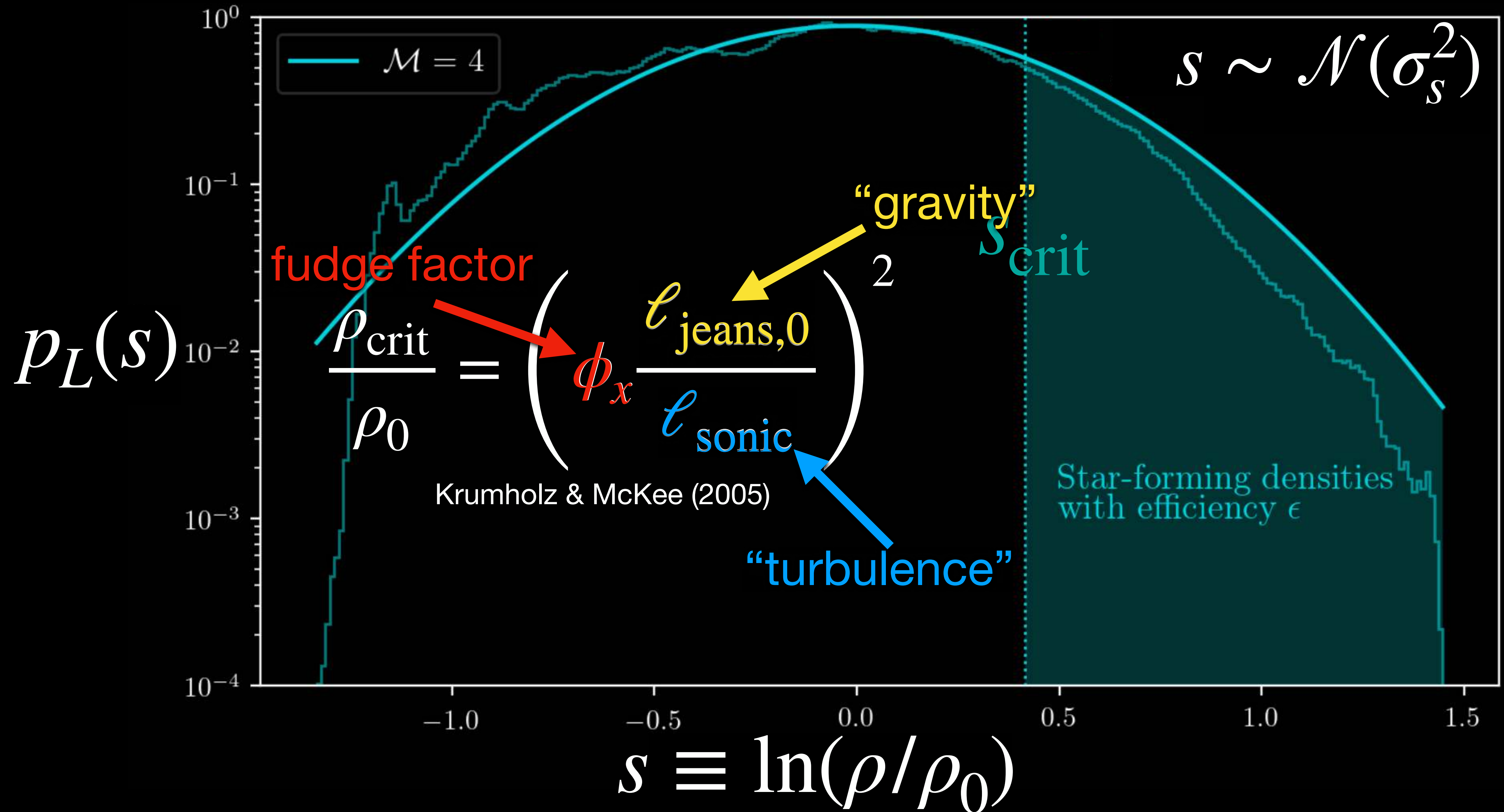
Turbulence-regulated star-formation based on $s \equiv \ln(\rho/\rho_0)$ statistics

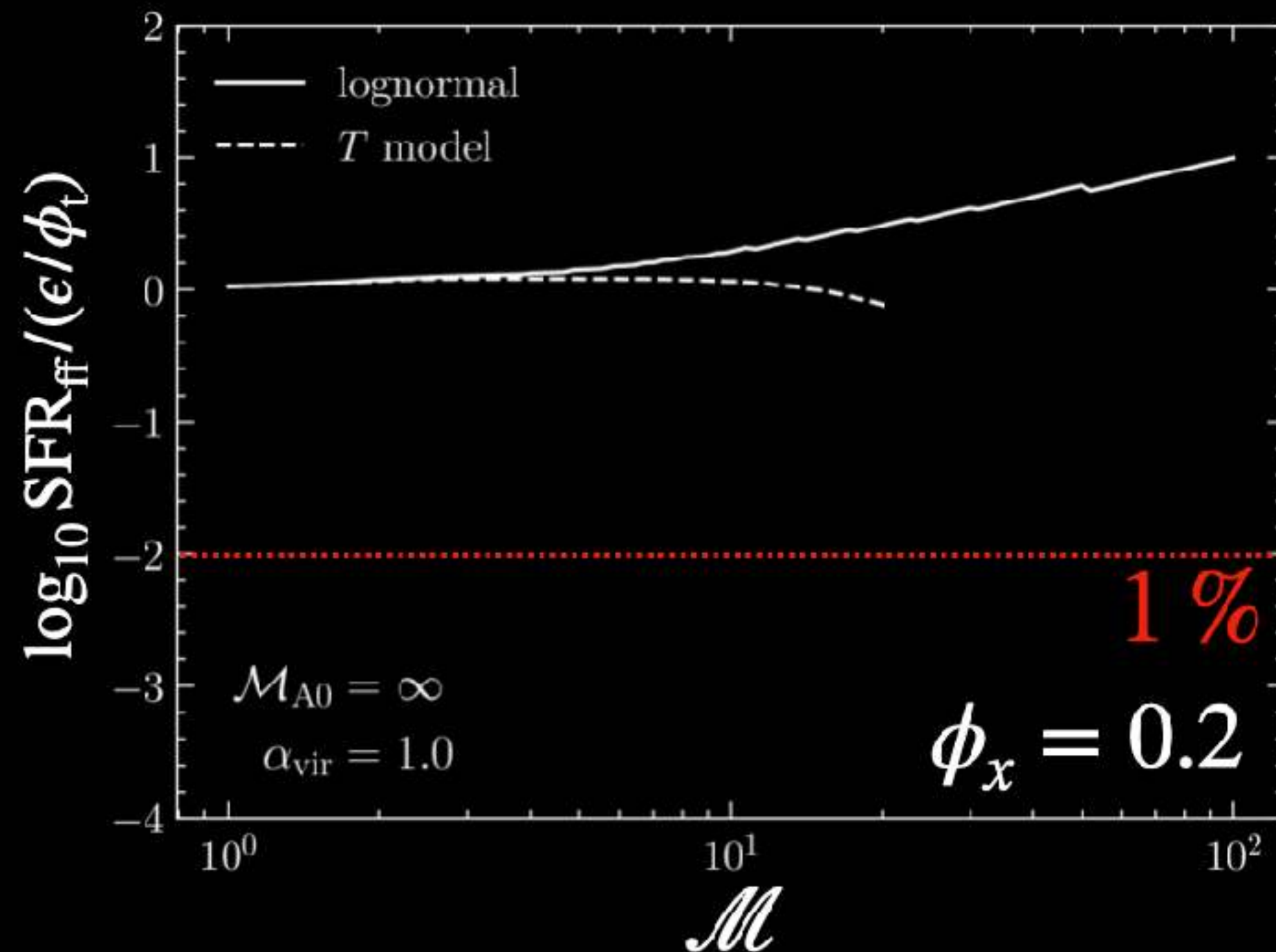
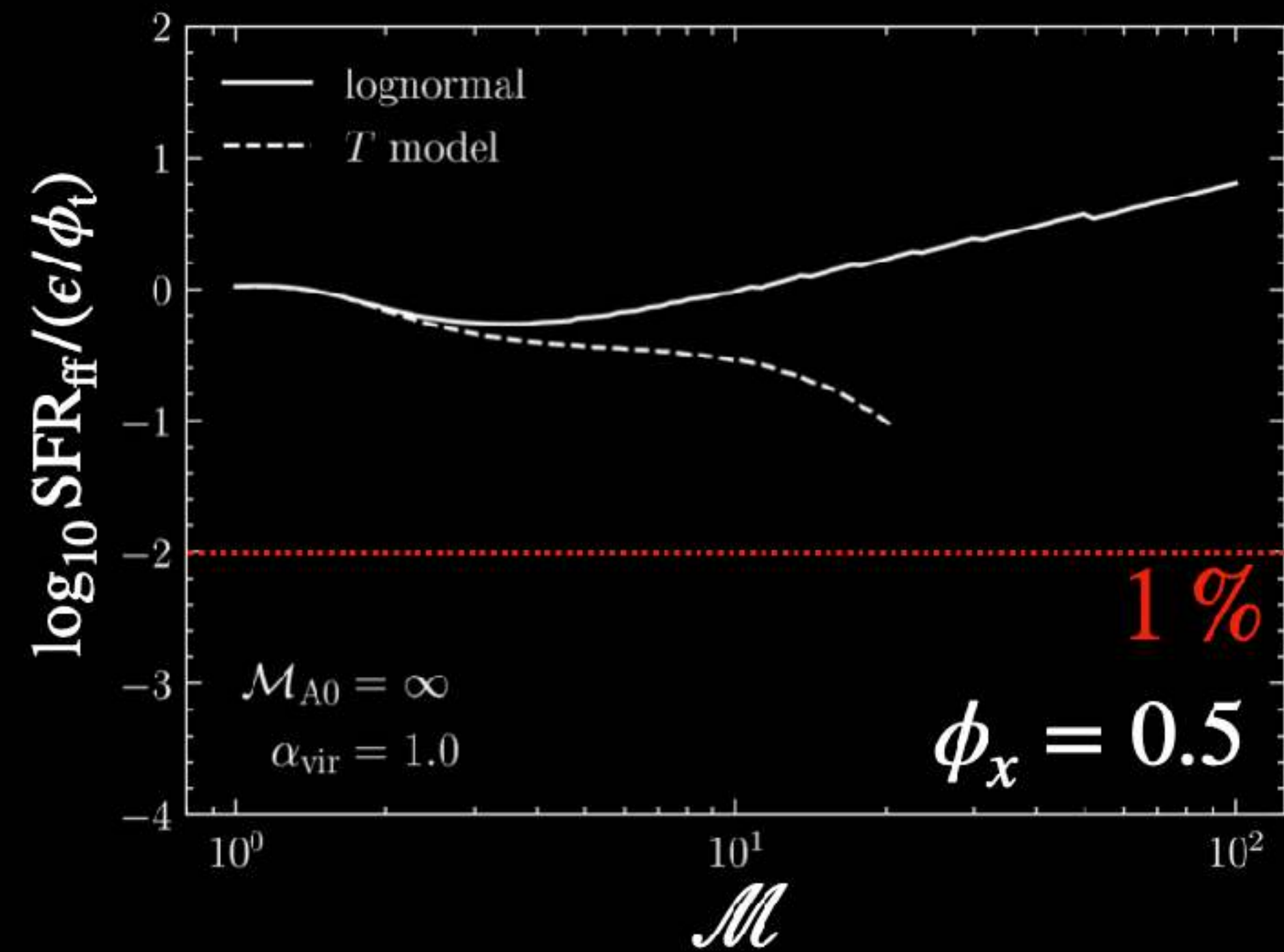
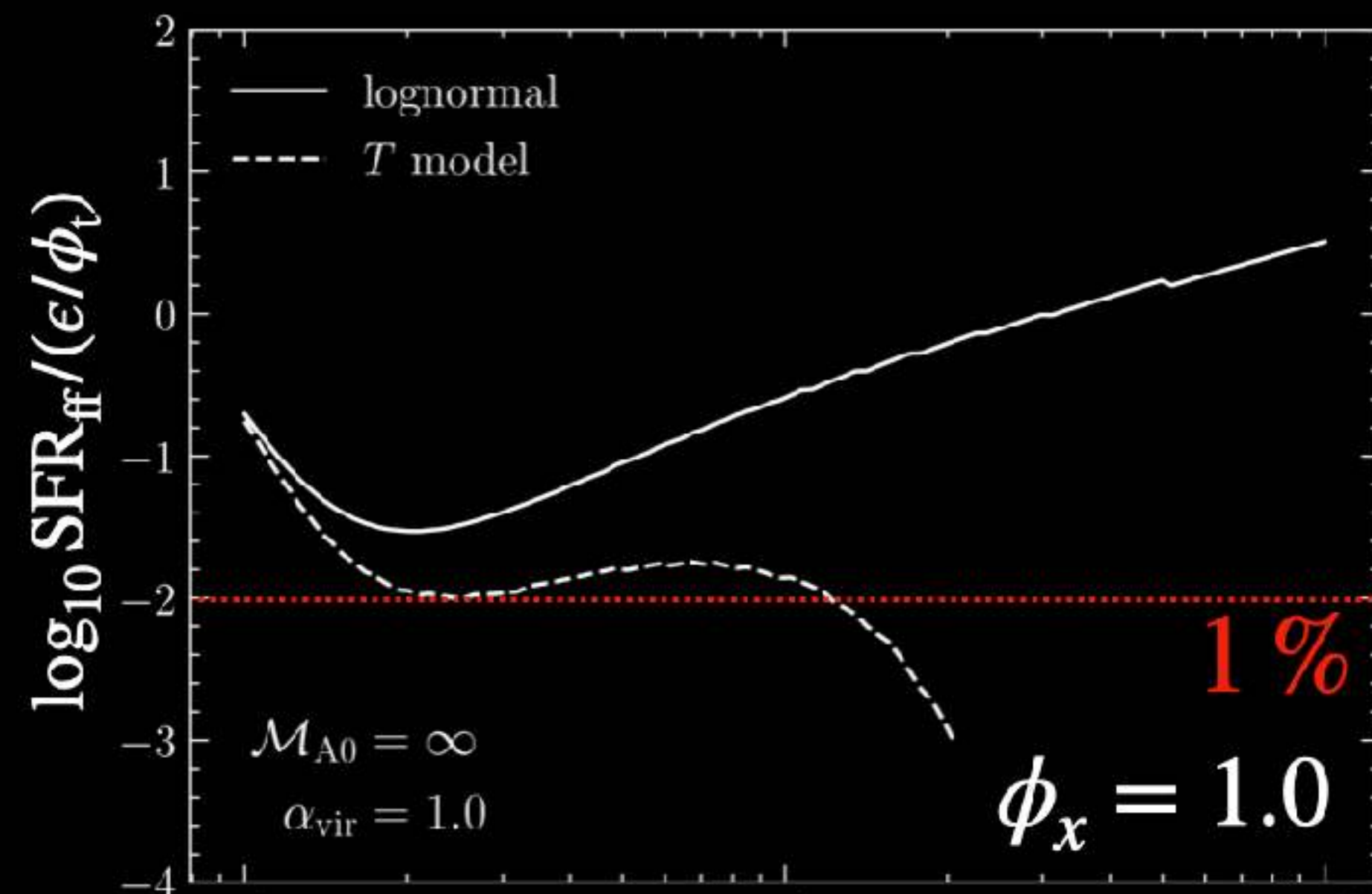
Krumholz & McKee (2005); Federrath & Klessen (2012); Mocz & Burkhart (2018), Burkhart & Mocz (2018)



Turbulence-regulated star-formation based on $s \equiv \ln(\rho/\rho_0)$ statistics

Krumholz & McKee (2005); Federrath & Klessen (2012); Mocz & Burkart (2018), Burkart & Mocz (2018)





In general SFR_{ff} is *very* sensitive to the value used for ϕ_x !!!

Be careful if setting $\phi_x \sim 1$!!

Magnetosonic scale

Federrath & Klessen (2012)

$$\mathcal{M} = \frac{\langle \delta v^2 \rangle_{\mathcal{V}}^{1/2}}{c_s}$$

$$\beta = \frac{c_s^2}{v_A^2}$$

L

$$P_{\text{therm}} + P_{\text{mag}}$$



$$\ell_{\text{ms}} = L(1 + \beta^{-1})\mathcal{M}^{-2}$$

“But what about magnetic fields?”

MHD energy cascade

Magneto-supersonic
cascade

$$E(k) \sim k^{-2}$$

maybe...

Magneto-subsonic
cascade

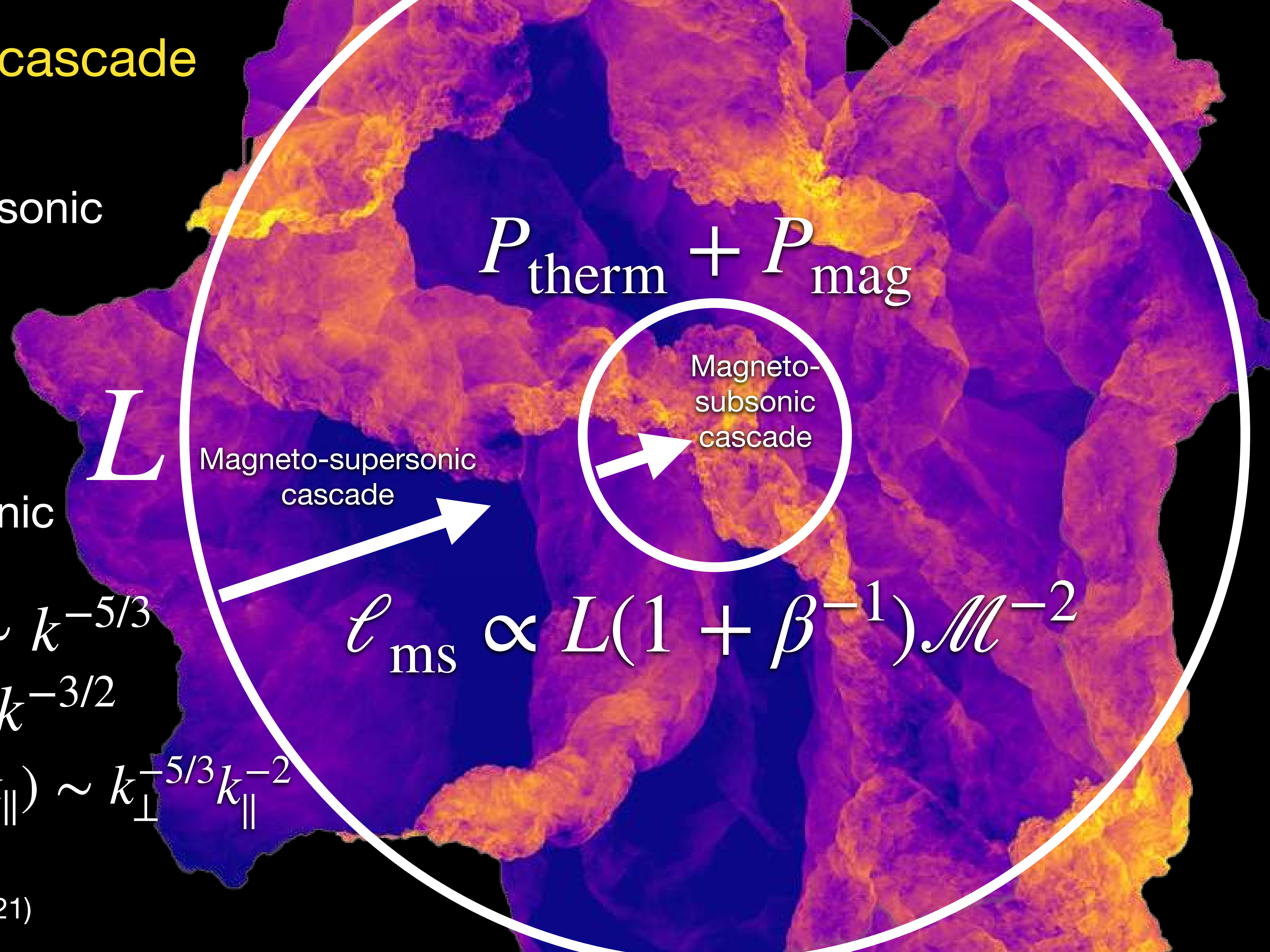
$$K41 : E(k) \sim k^{-5/3}$$

$$IK : E(k) \sim k^{-3/2}$$

$$GS95 : E(k_{\perp}, k_{\parallel}) \sim k_{\perp}^{-5/3} k_{\parallel}^{-2}$$

⋮

Schekochihin (2021)



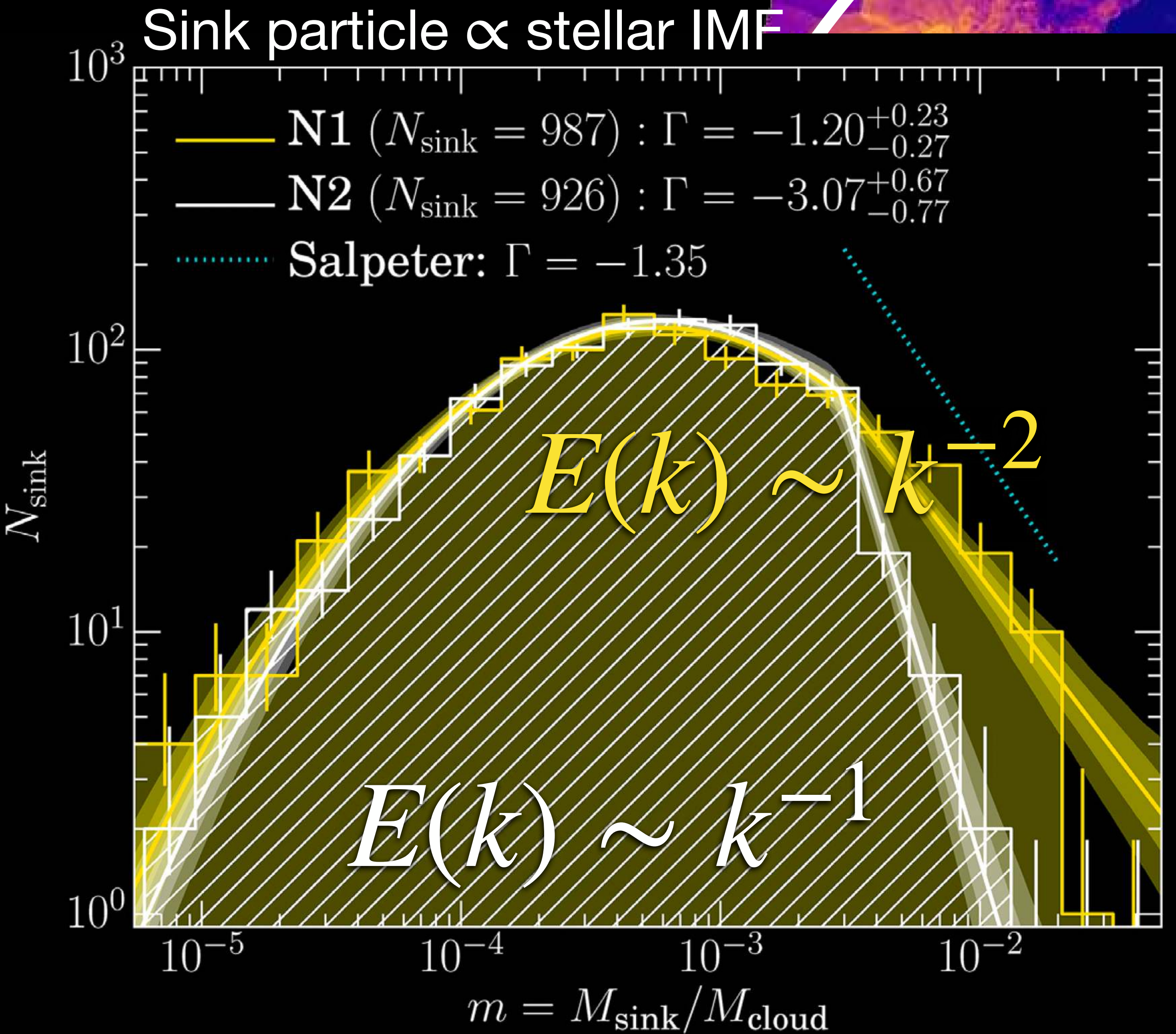
$$P_{\text{therm}} + P_{\text{mag}}$$

Magneto-subsonic
cascade

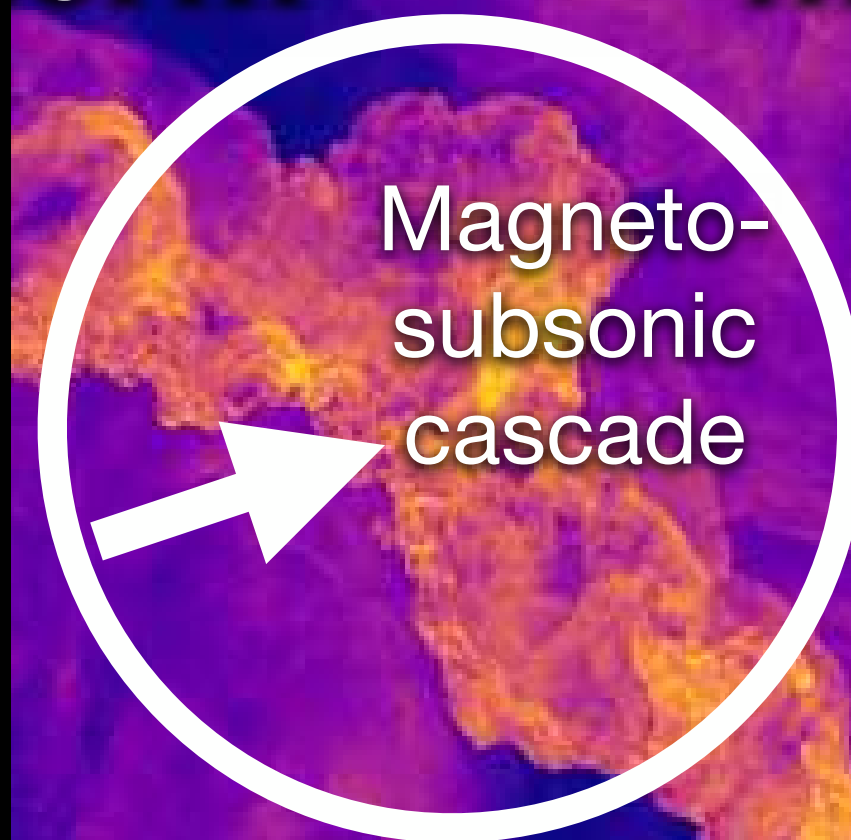
Magneto-supersonic
cascade

$$\ell_{\text{ms}} \propto L(1 + \beta^{-1}) \mathcal{M}^{-2}$$

Why should astrophysicists care?

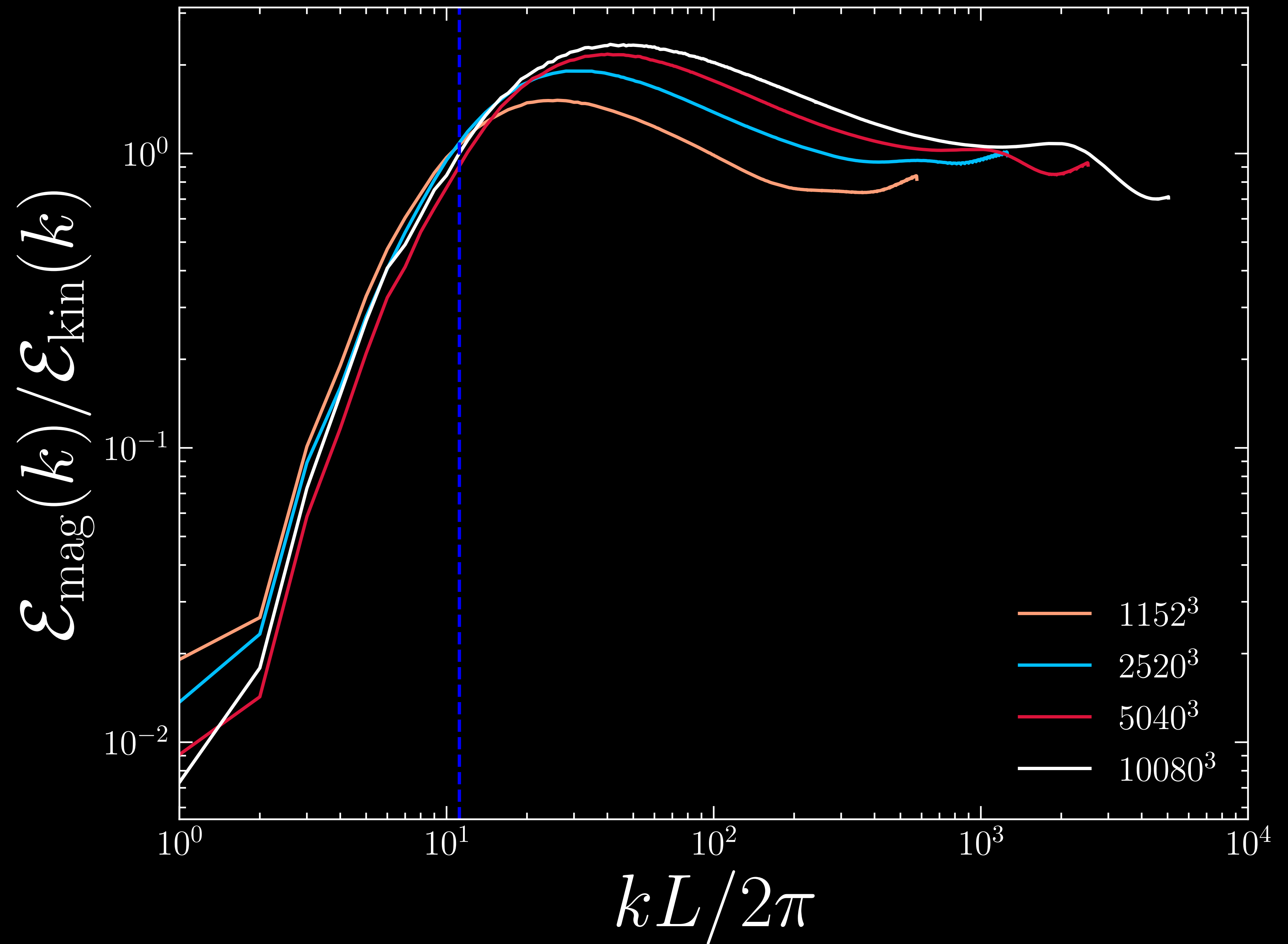


term + P_{mag}

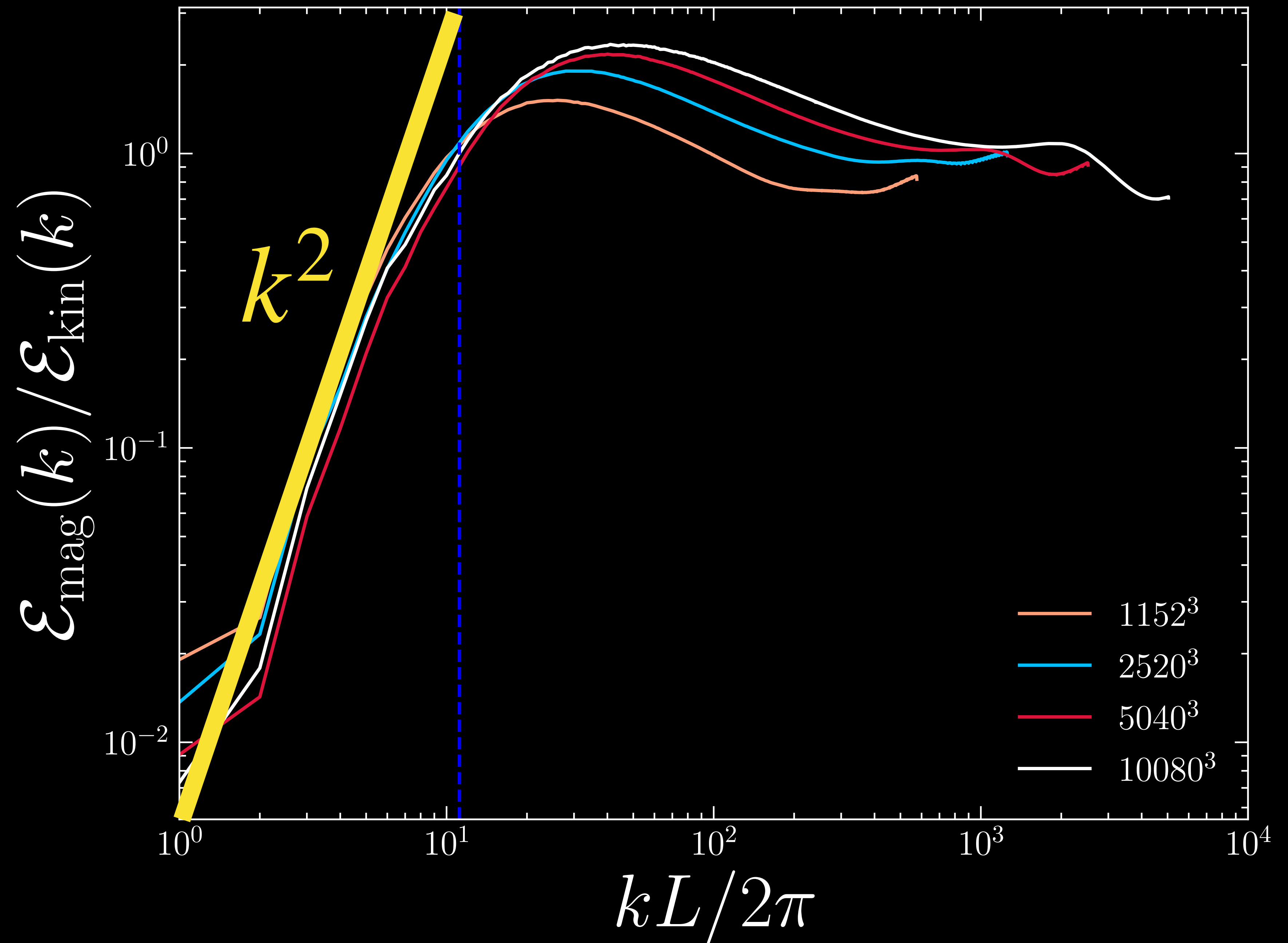


More energy on the small-scales (more small-scale support) implies less high-mass stars

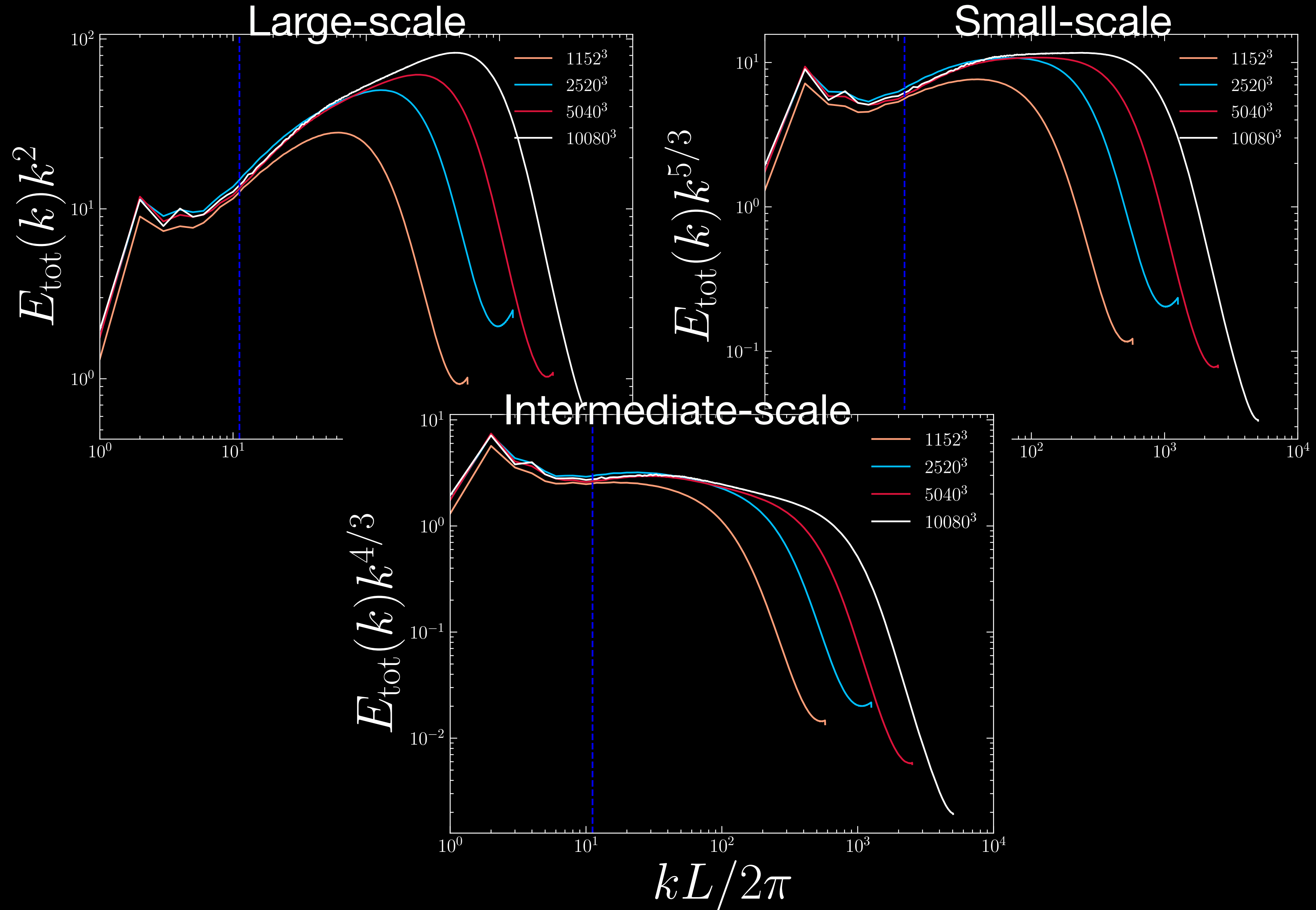
Energy ratio



Energy ratio



Total energy cascade

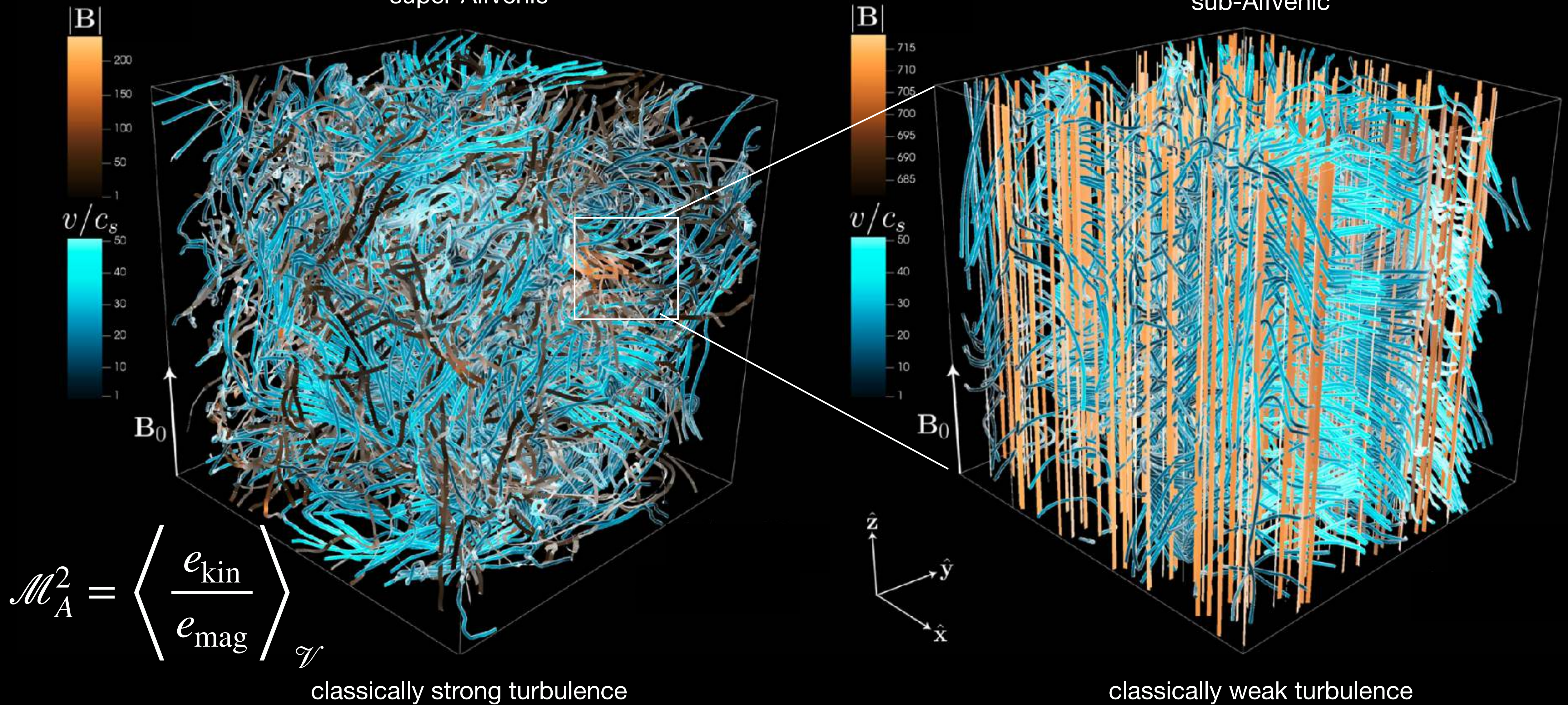


$$\mathcal{M}_A > 1$$

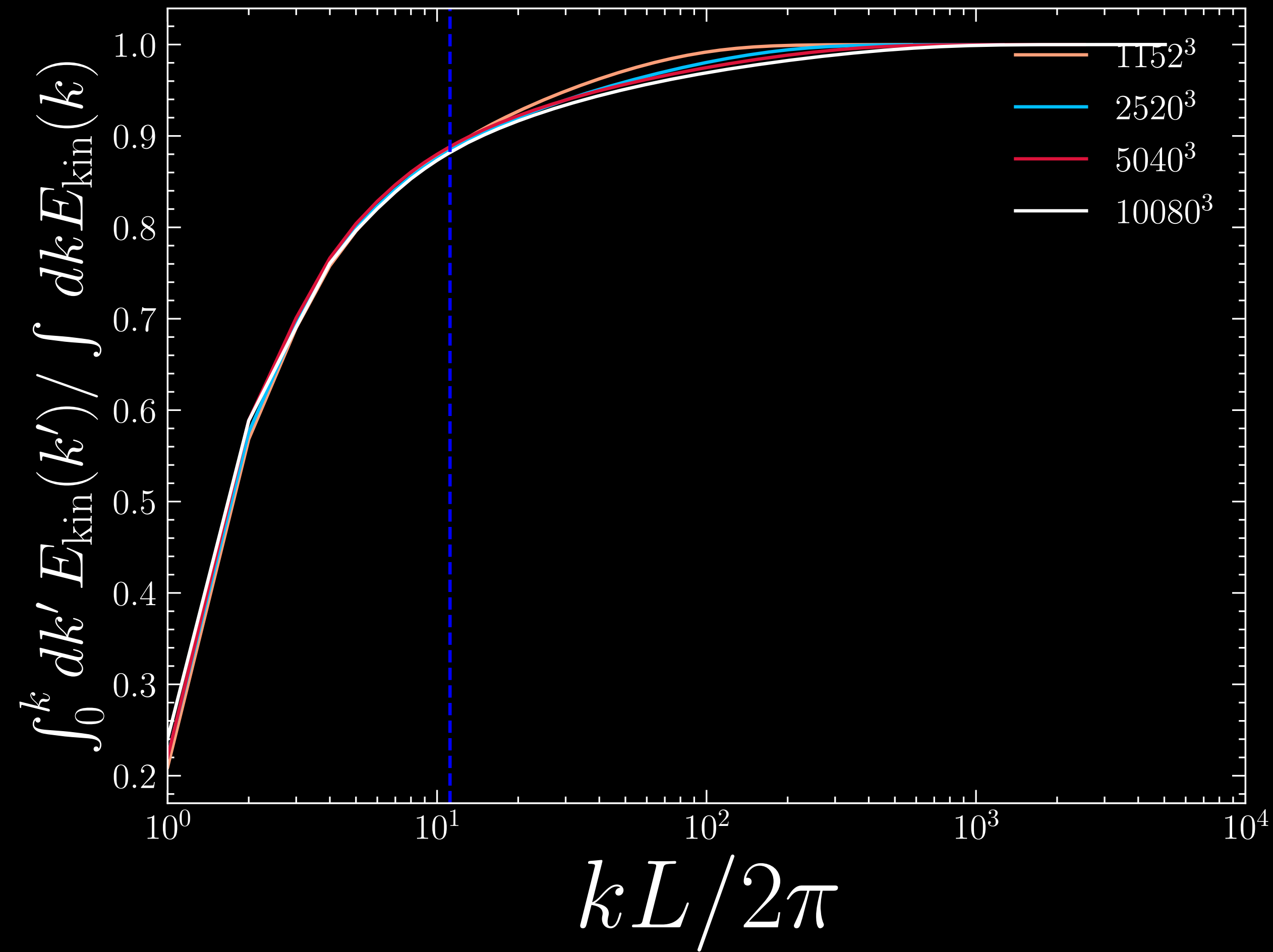
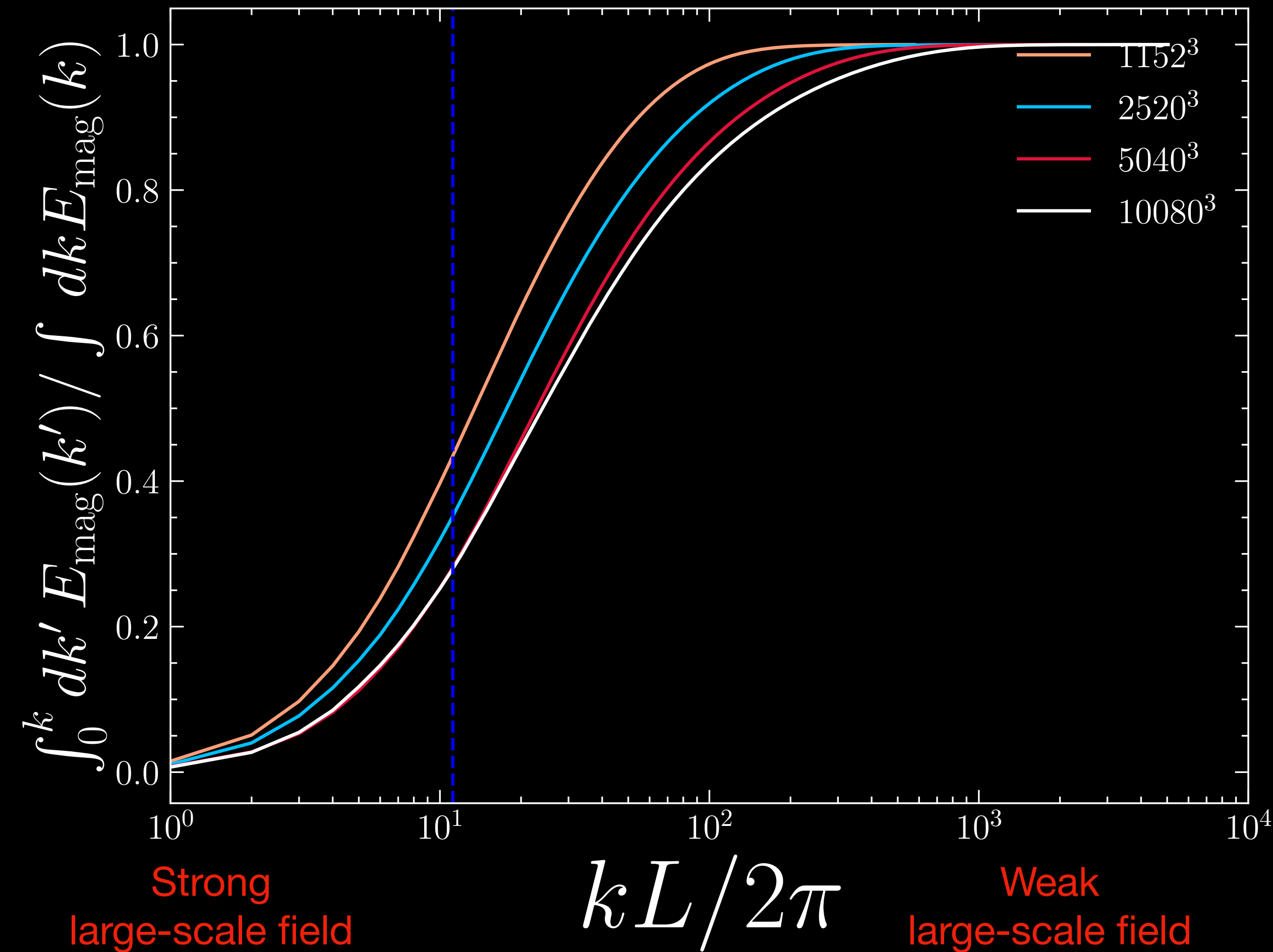
super-Alfvénic

$$\mathcal{M}_A < 1$$

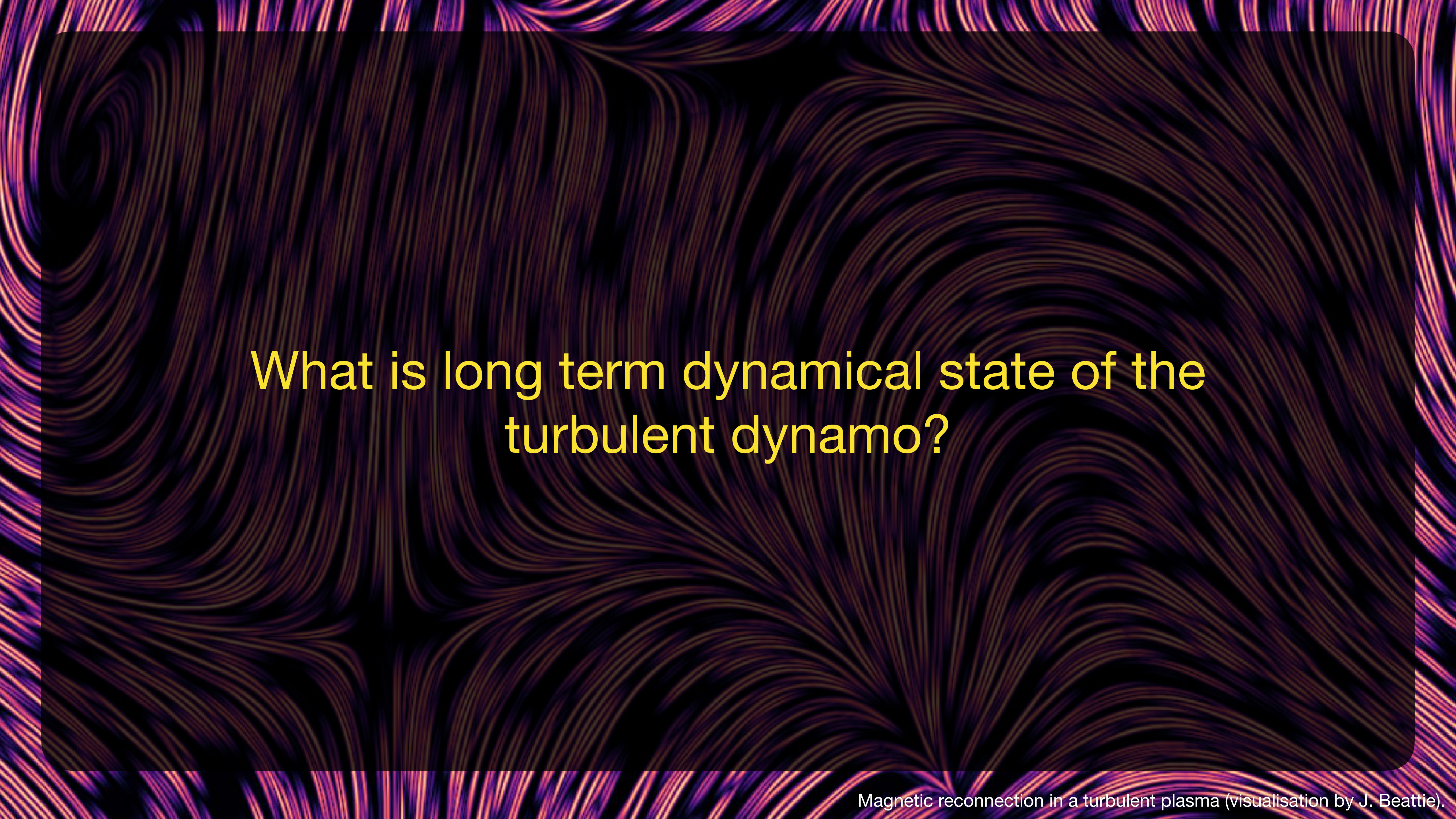
sub-Alfvénic



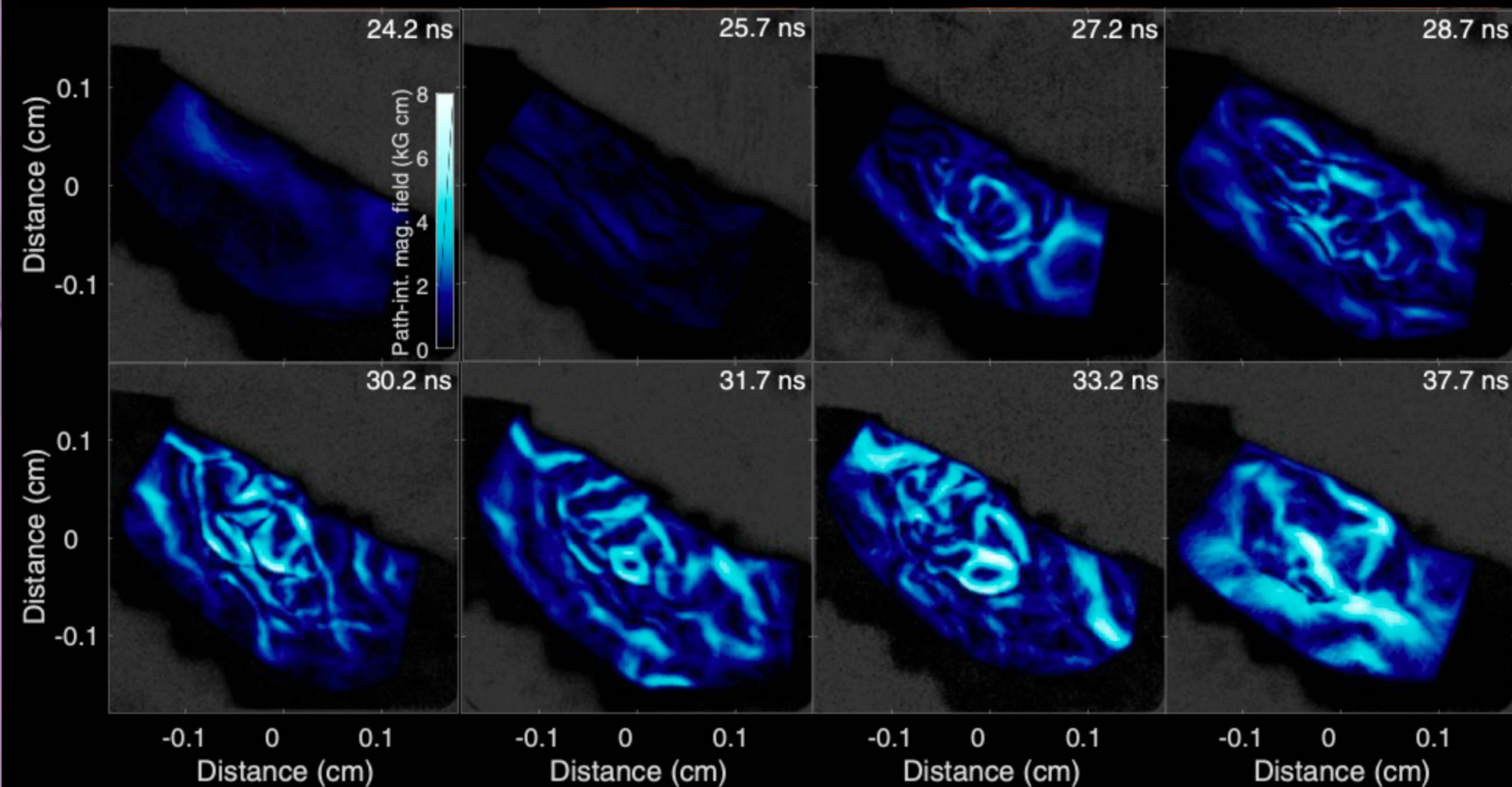
Sub-Alfvénic turbulence is embedded inside of super-Alfvénic



turbulent magnetic fields are intrinsically small-scale (compared to velocity)!

The background of the slide is a complex, abstract pattern of swirling, wavy lines in shades of purple, blue, and magenta. The lines are dense and create a sense of motion and depth, resembling a turbulent flow or a magnetic field visualization. The overall effect is a vibrant, textured backdrop for the text.

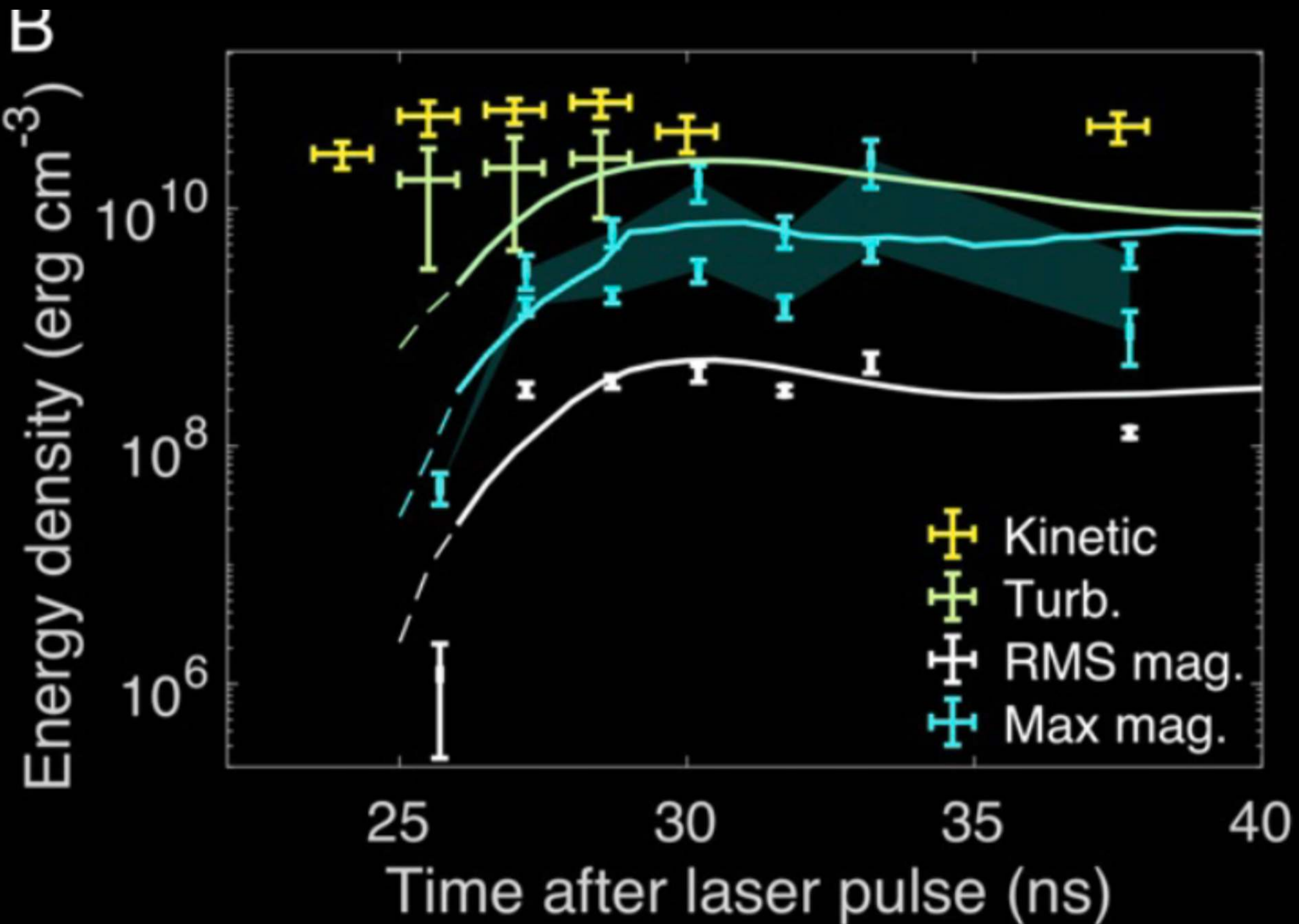
What is long term dynamical state of the
turbulent dynamo?



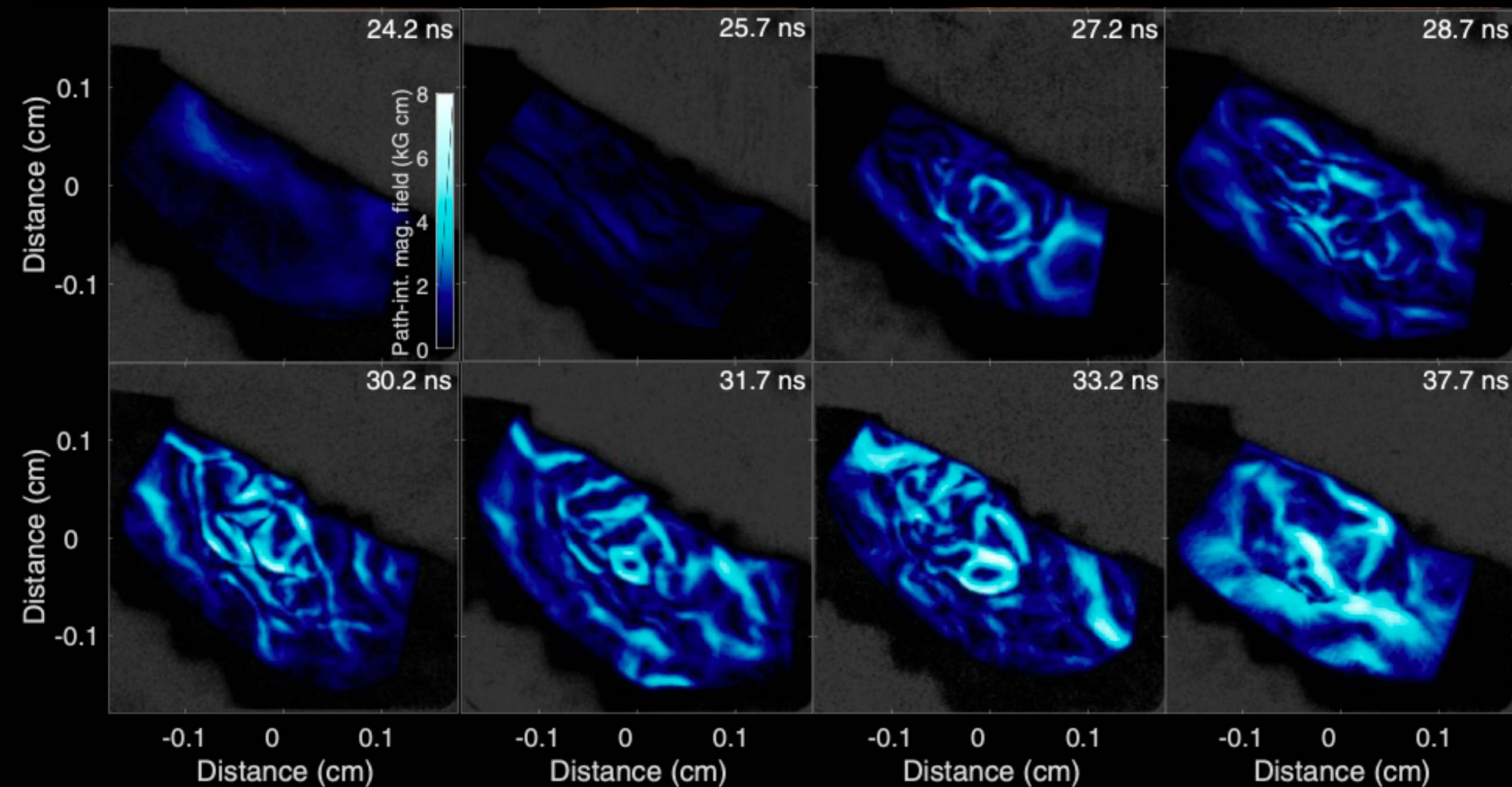
Time

The turbulent dynamo

magnetic fields grow in the presence of turbulence



Path integrated magnetic field from
proton imaging



Time



The turbulent dynamo

⇒ magnetic fields grow in the presence of ISM turbulence

straight, strong fields ($\kappa \propto B^{-1/2}$; Schekochihin+2004) penetrating through Orion A
Orion A in infrared; ESA/Herschel/Planck; J. D. Soler, MPIA

The turbulent dynamo

three stage growth

1) $\frac{dE_{\text{mag}}}{dt} = \gamma_1 E_{\text{mag}}$ (kinematic stage)

2) $\frac{dE_{\text{mag}}}{dt} = \gamma_2$ (non-linear stage)

3) $\frac{dE_{\text{mag}}}{dt} = 0$ (saturation stage)

The turbulent dynamo

three stage growth

Initial conditions

1) $\frac{dE_{\text{mag}}}{dt} = \gamma_1 E_{\text{mag}},$

$$E_{\text{mag},0} \ll 1$$

2) $\frac{dE_{\text{mag}}}{dt} = \gamma_2,$

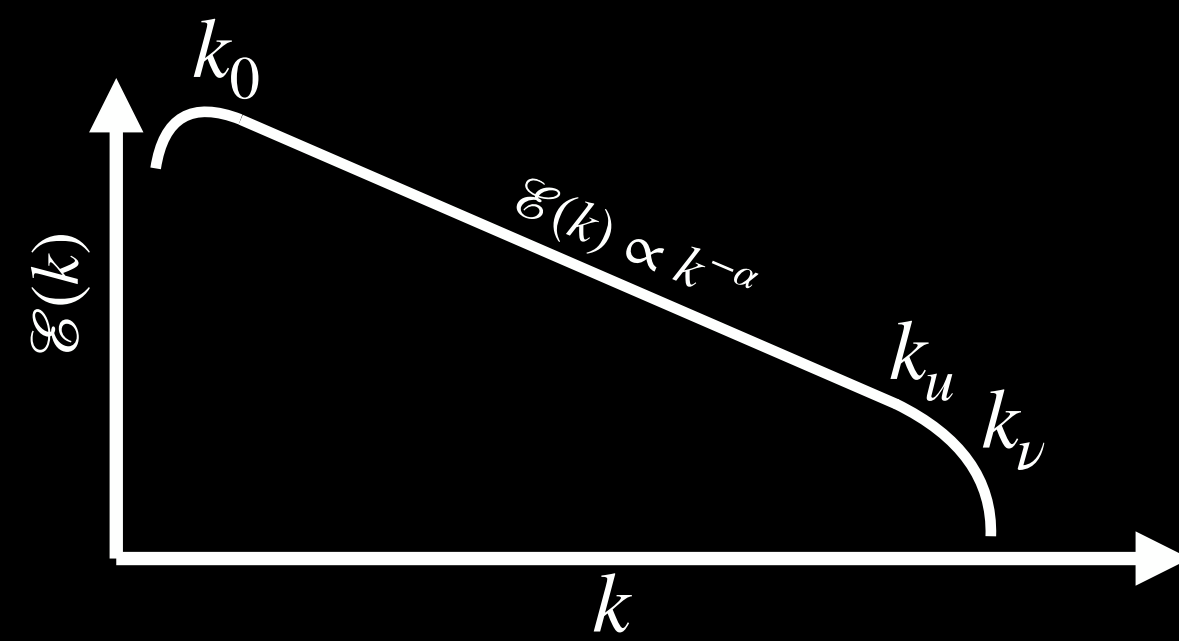
$$E_{\text{mag},0} \leftarrow \frac{\nabla \cdot (\rho v \otimes v)}{(\nabla \times b) \times b} \sim 1$$

3) $\frac{dE_{\text{mag}}}{dt} = 0,$

$$E_{\text{mag},0} \leftarrow \frac{E_{\text{mag}}}{E_{\text{kin}}} \sim 1$$

Turbulence

Cascade



until

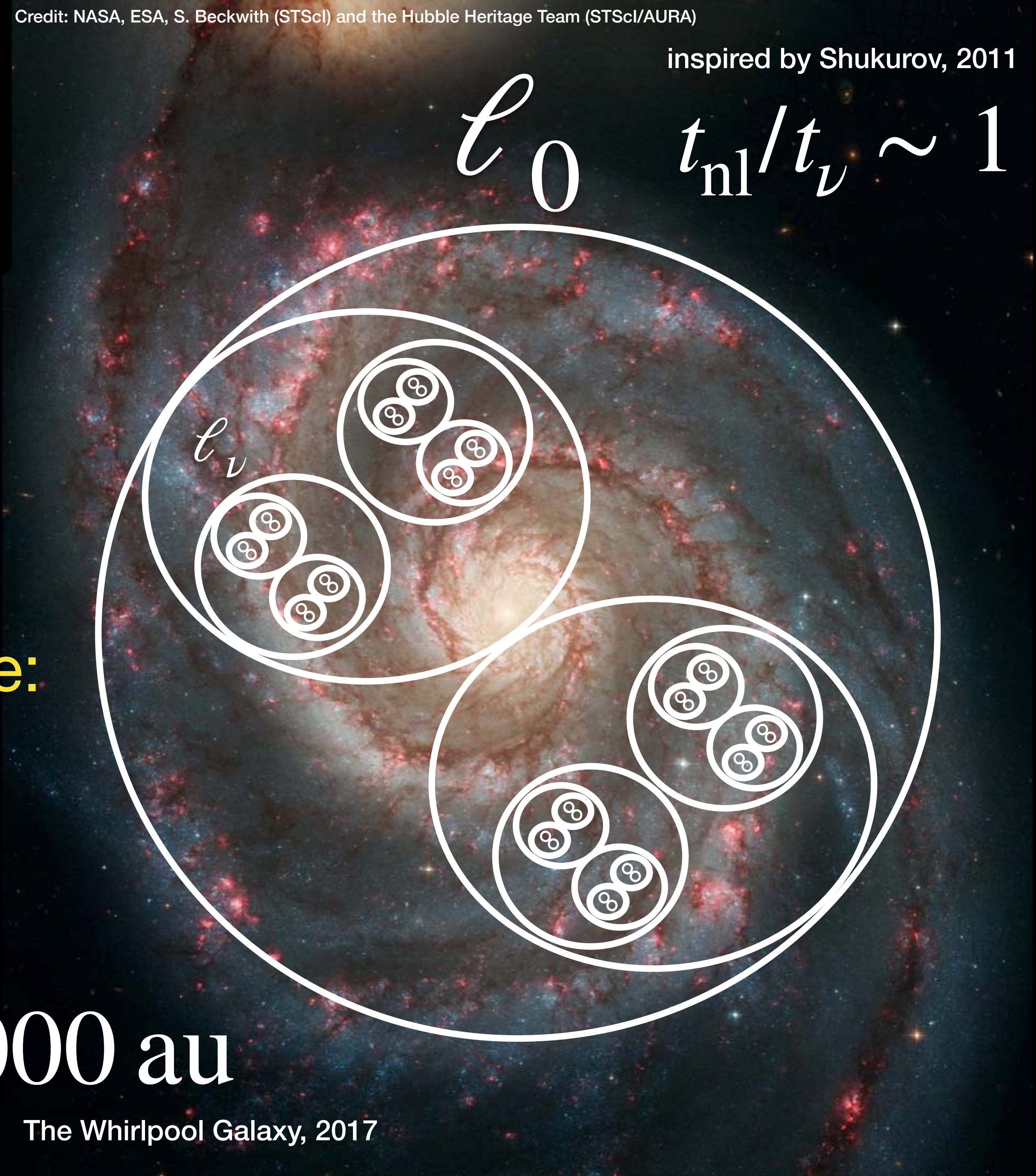
$$\text{Re} = \frac{|\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v})|}{|2\nu \nabla \cdot (\rho \mathbf{S})|} = 1$$

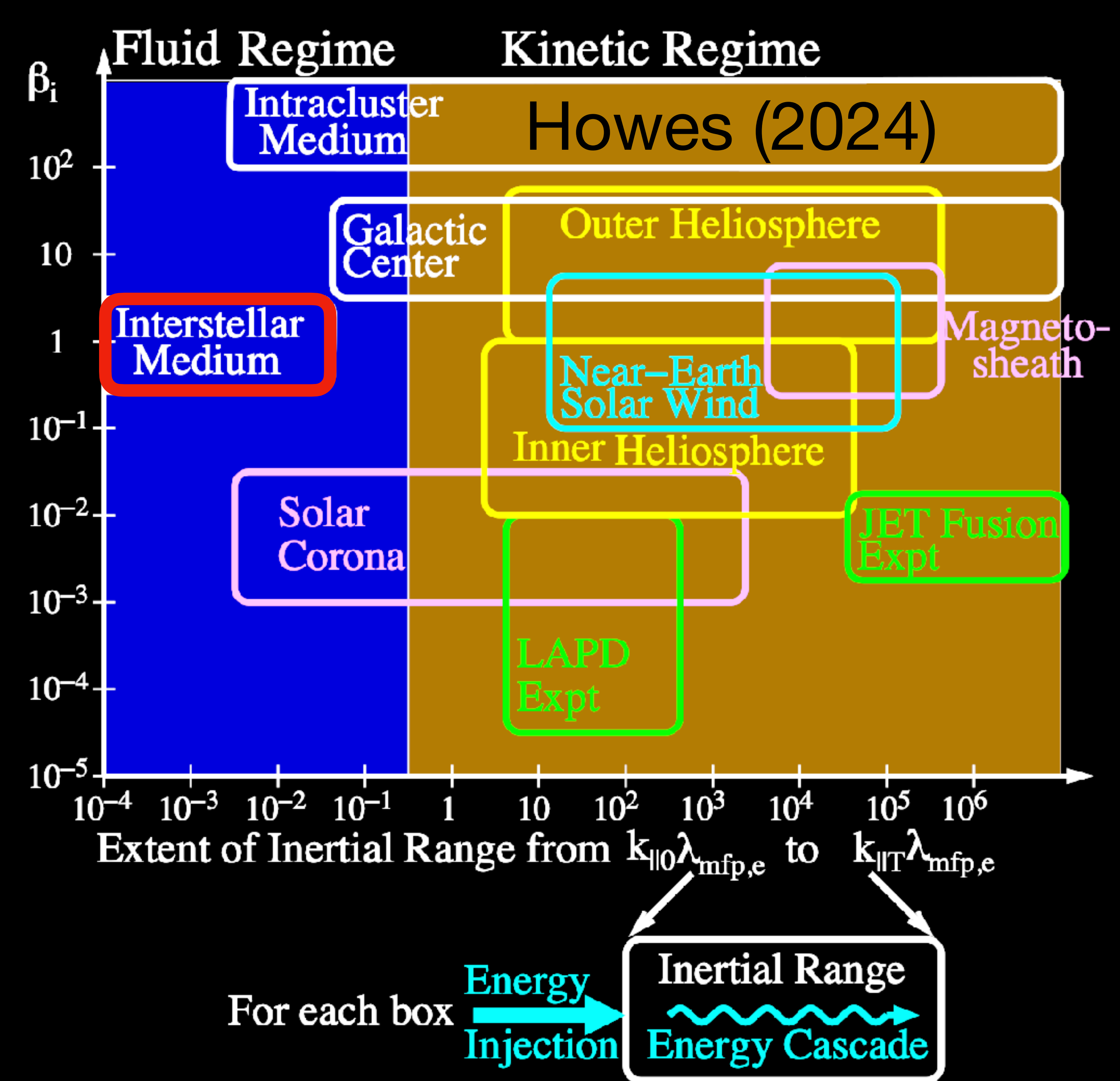
dissipation scales for WIM cascade:

$$\ell_\nu \sim \text{Re}^{-3/4} \ell_0$$

$$\text{WIM: } \text{Re} \sim 10^7$$

$$\ell_0 \sim 1 \text{ kpc} \implies \ell_\nu \sim 1000 \text{ au}$$





Stochastically forced compressible MHD equations (ILES)

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\partial_t (\rho \mathbf{u}) + \nabla \cdot \left(\rho \mathbf{u} \otimes \mathbf{u} + p \mathbb{I} - \frac{1}{4\pi} \mathbf{b} \otimes \mathbf{b} \right) = \boxed{\rho \mathbf{f}} + \nabla \cdot \mathbb{D}_\nu(\rho \mathbf{u})$$

$$\partial_t \mathbf{b} + \nabla \cdot (\mathbf{u} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{u}) = \nabla \cdot \mathbb{D}_\eta(\mathbf{b})$$

$$\nabla \cdot \mathbf{b} = 0$$

the turbulence source function

$$p = c_s^2 \rho + \frac{1}{8\pi} \mathbf{b} \cdot \mathbf{b}$$

Stochastically forced compressible MHD equations (ILES)

$$d\hat{\mathbf{f}}(\mathbf{k}, t) = f_0(\mathbf{k})\mathbb{P}(\mathbf{k}) \cdot d\mathbf{W}(t) - \hat{\mathbf{f}}(\mathbf{k}, t)\frac{dt}{t_0}$$

$d\mathbf{W}(t)$ Wiener process that draws delta correlated from $\sim \mathcal{N}(0,1)$

k space projection tensor

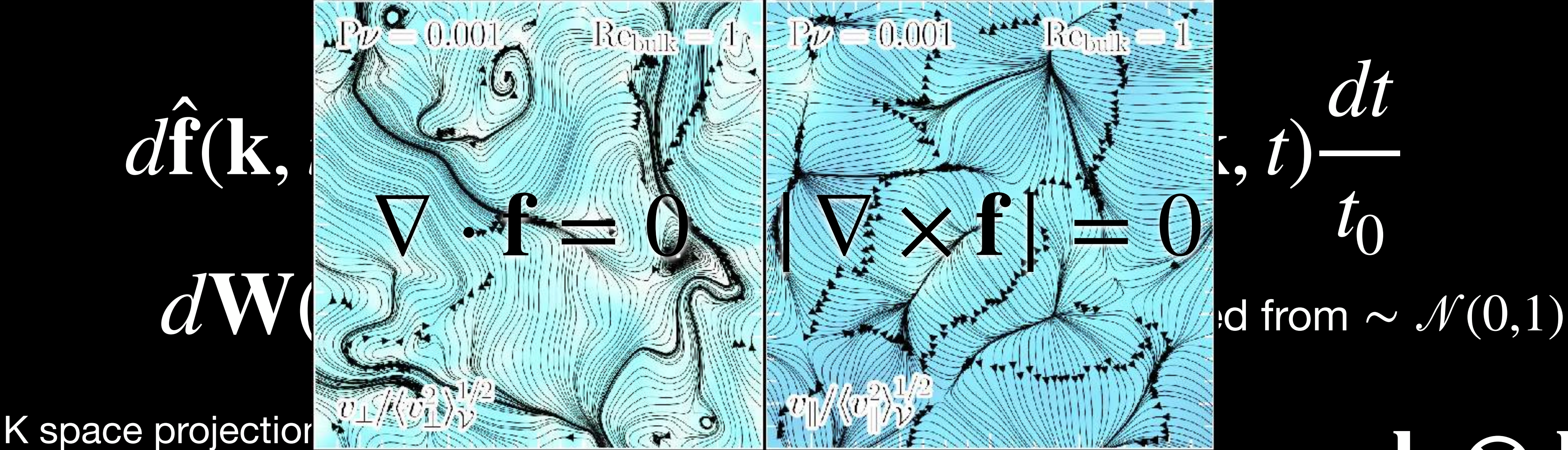
$$\mathbb{P} = \zeta\mathbb{P}^\perp + (1 - 2\zeta)\mathbb{P}^\parallel = \zeta\mathbb{I} + (1 - 2\zeta)\frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2}$$

$$\zeta = 1 \implies \nabla \cdot \mathbf{f} = 0$$

$$\zeta = 0 \implies \nabla \times \mathbf{f} = 0$$

t_0 e-folding time of the forcing / correlation time / outer-scale turbulent turnover time

Stochastically forced compressible MHD equations (ILES)



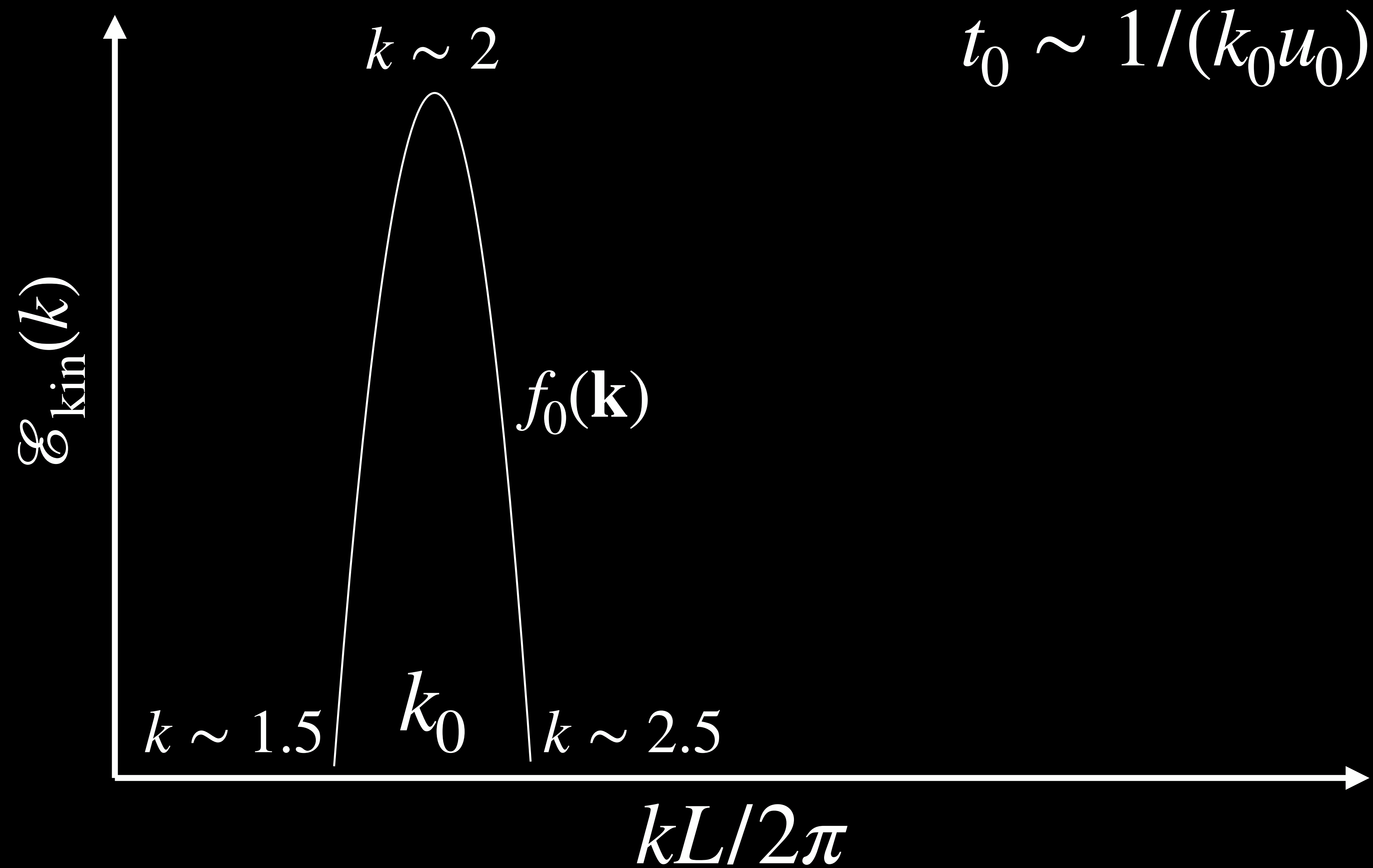
$$\mathbb{P} = \zeta \mathbb{P}^\perp + (1 - 2\zeta) \mathbb{P}^\parallel = \zeta \mathbb{I} + (1 - 2\zeta) \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2}$$

$$\zeta = 1 \implies \nabla \cdot \mathbf{f} = 0$$

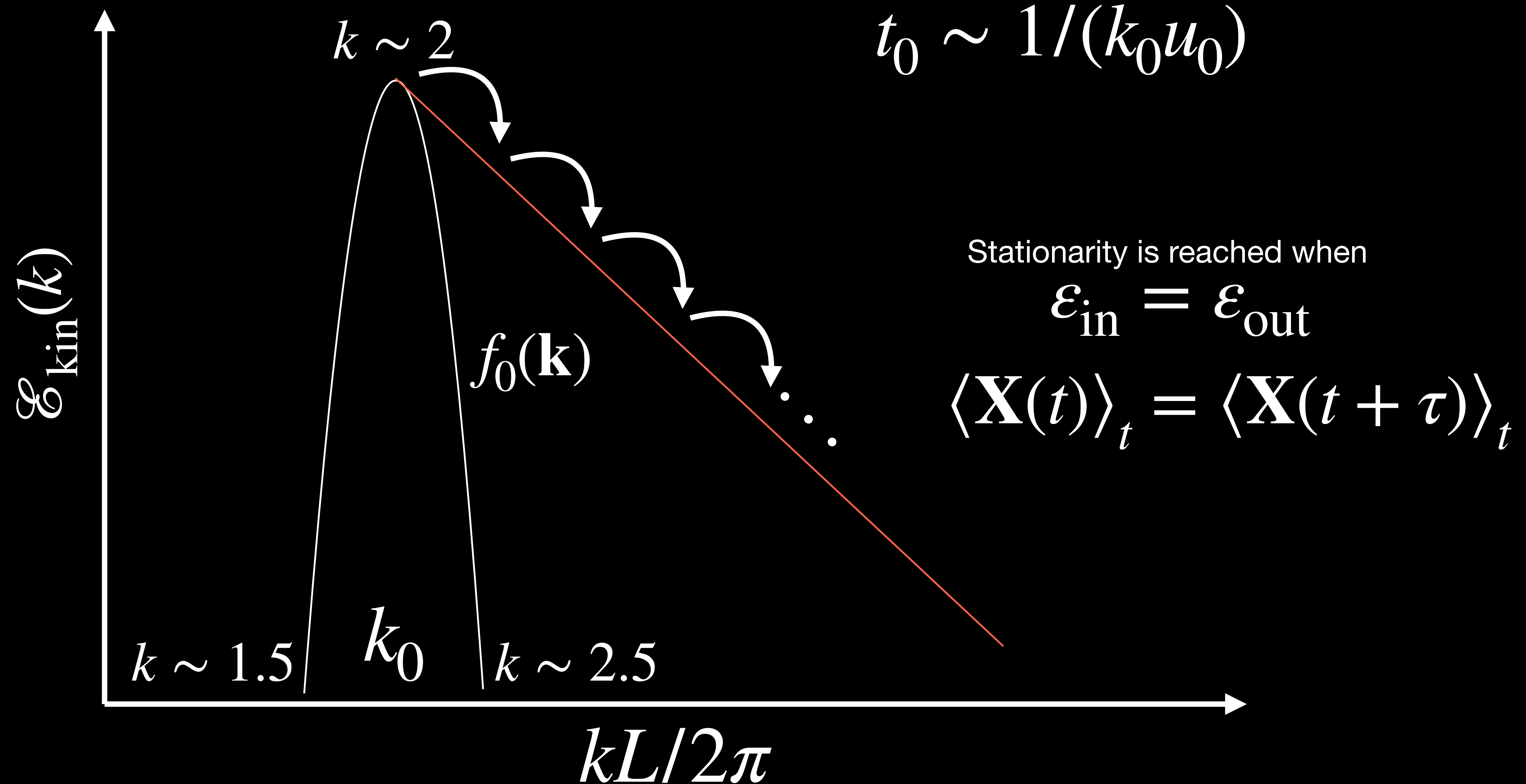
$$\zeta = 0 \implies \nabla \times \mathbf{f} = 0$$

t_0 e-folding time of the forcing / correlation time / outer-scale turbulent turnover time

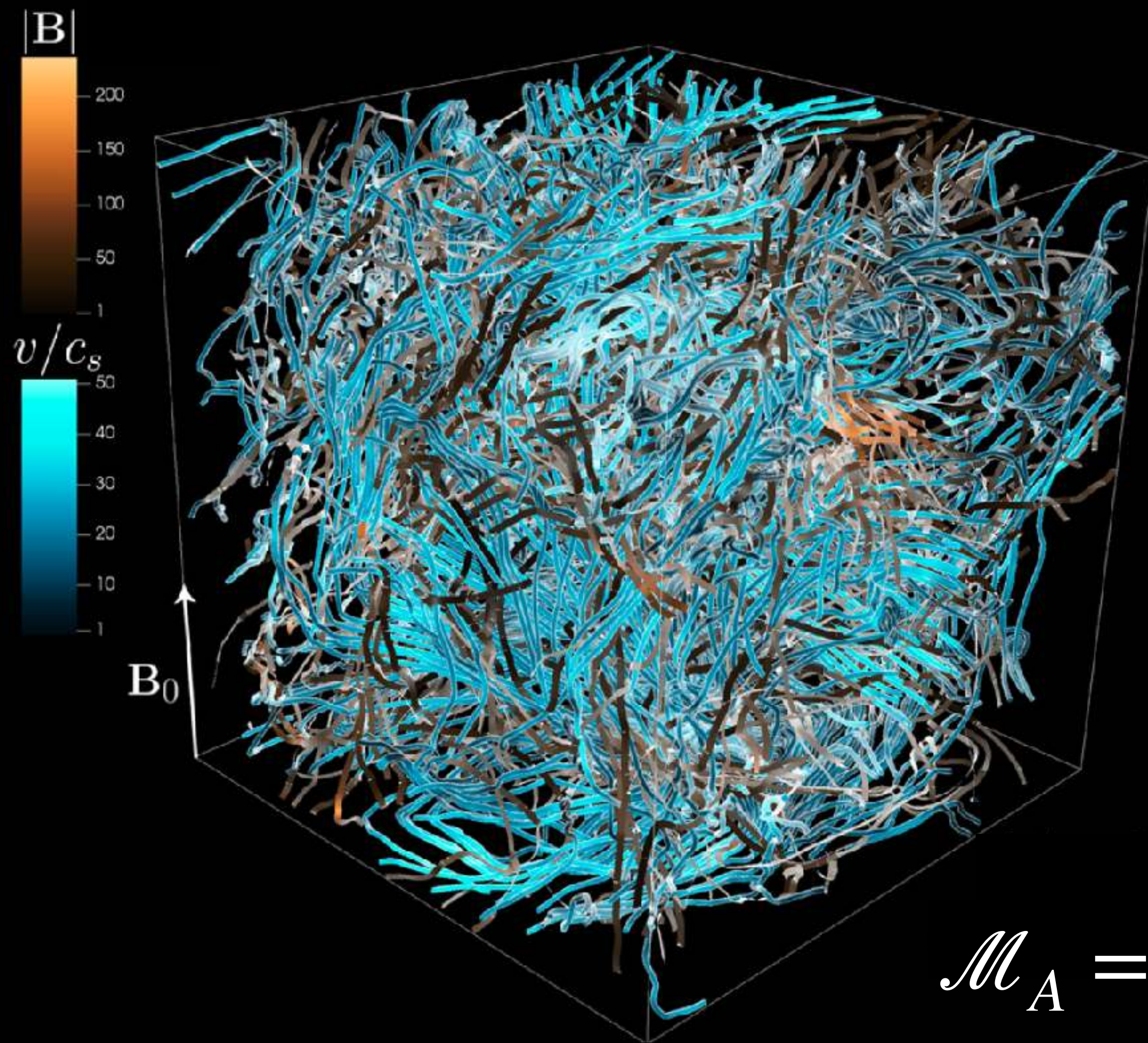
Stochastically forced Compressible MHD equations (ILES)



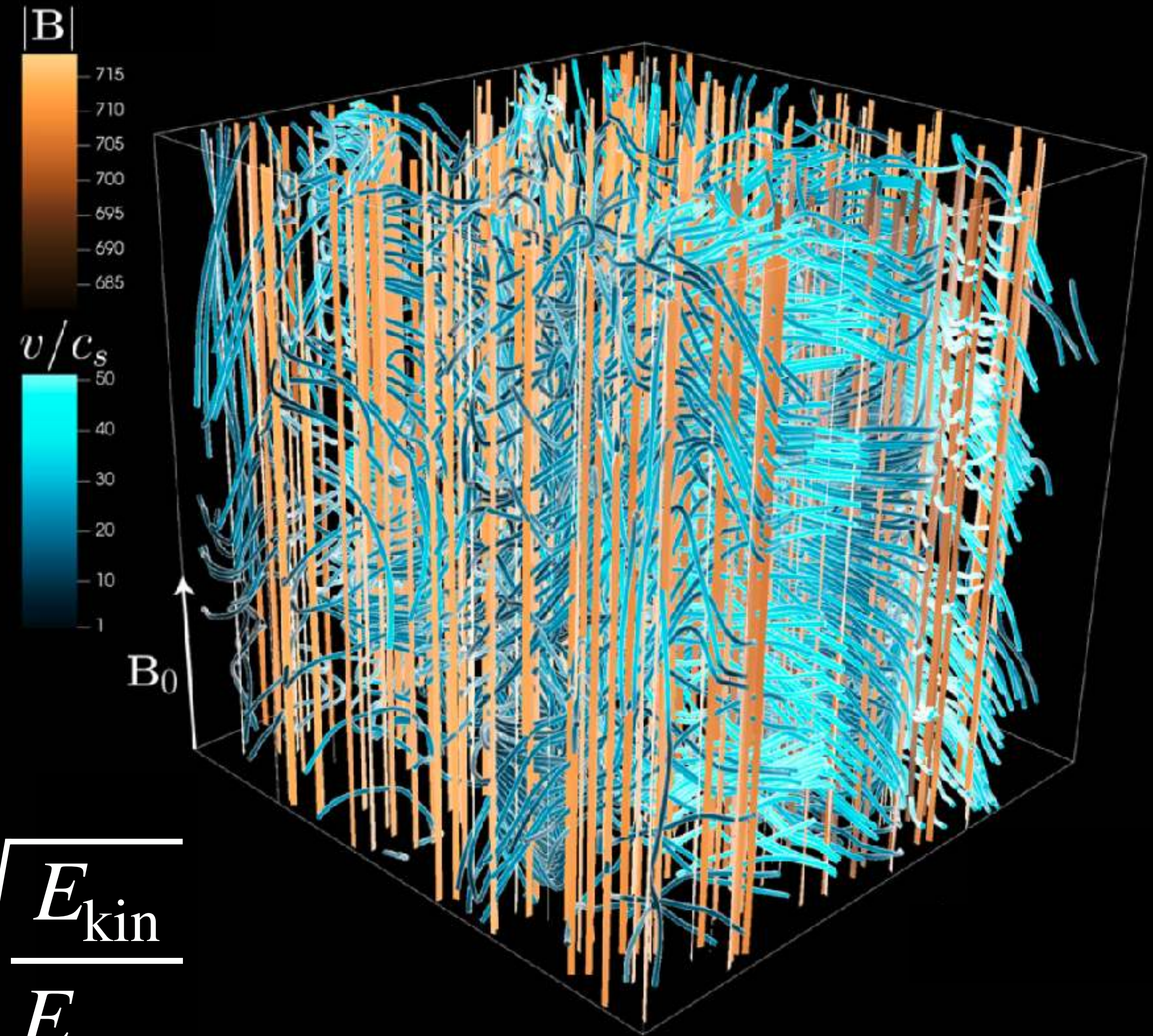
Stochastically forced Compressible MHD equations (ILES)



$$B_0 = 0$$



$$B_0 \gg b_{\text{turb}}$$



$$\mathcal{M}_A = \sqrt{\frac{E_{\text{kin}}}{E_{\text{mag}}}}$$