Aspects of magnetic growth & the turbulent dynamo

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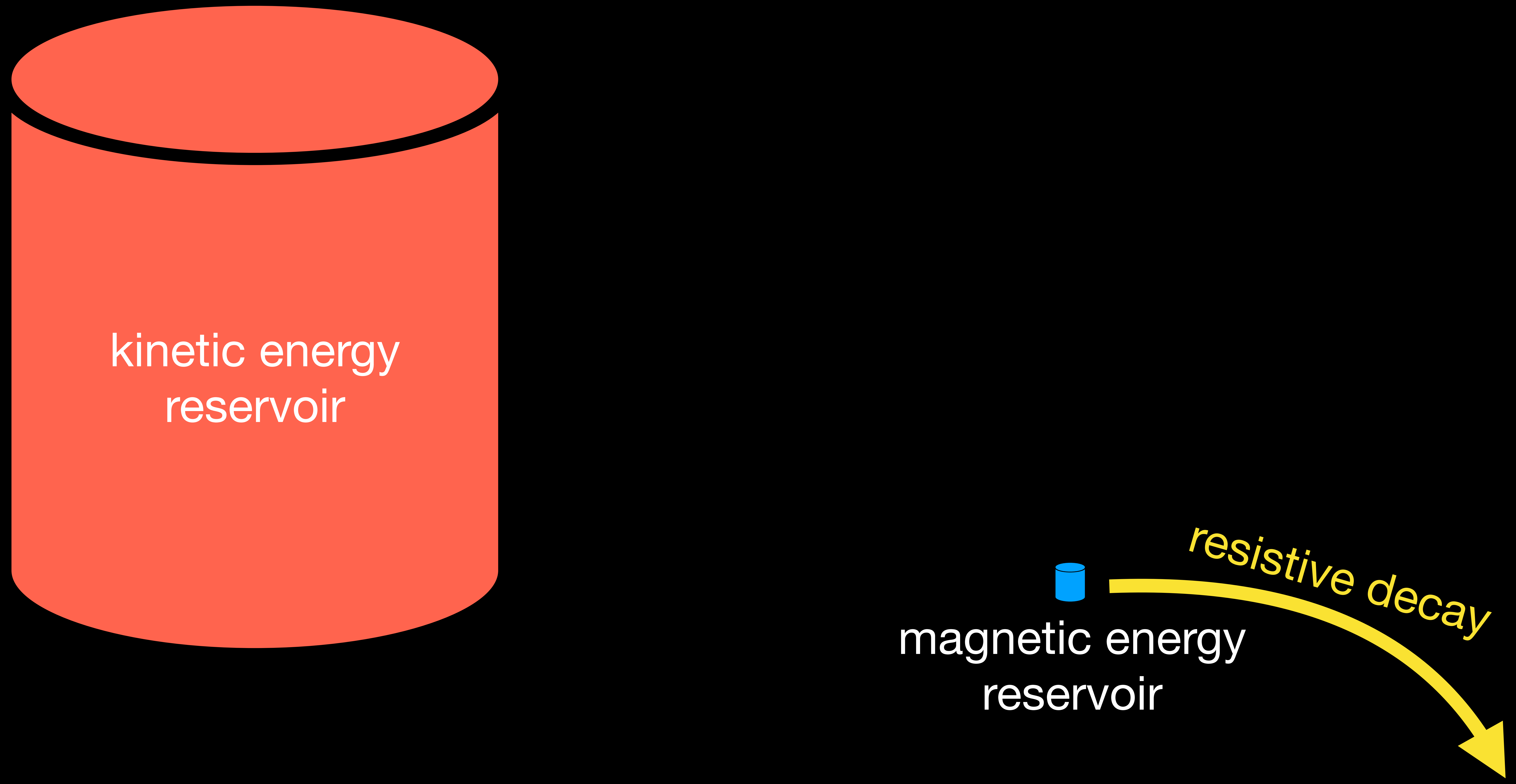
$L \sim \mathcal{O}(\text{kpc})$
M82 (Cigar Galaxy)
M51 (Whirlpool Galaxy)
small scale, disordered magnetic fields
$\mathcal{E}_{\text{kin}} \sim \mathcal{E}_{\text{mag}}$
Beck (2015)
$\mathbf{B} = \langle \mathbf{B} \rangle + \mathbf{b}$
large scale dynamo theory
small scale dynamo theory
large scale, ordered magnetic fields
Lopez-Rodriguez + (2021)
Credit: NASA, the SOFIA science team, A. Borlaff; NASA, ESA, S. Beckwith (STScI) and the Hubble Heritage Team (STScI/AURA)

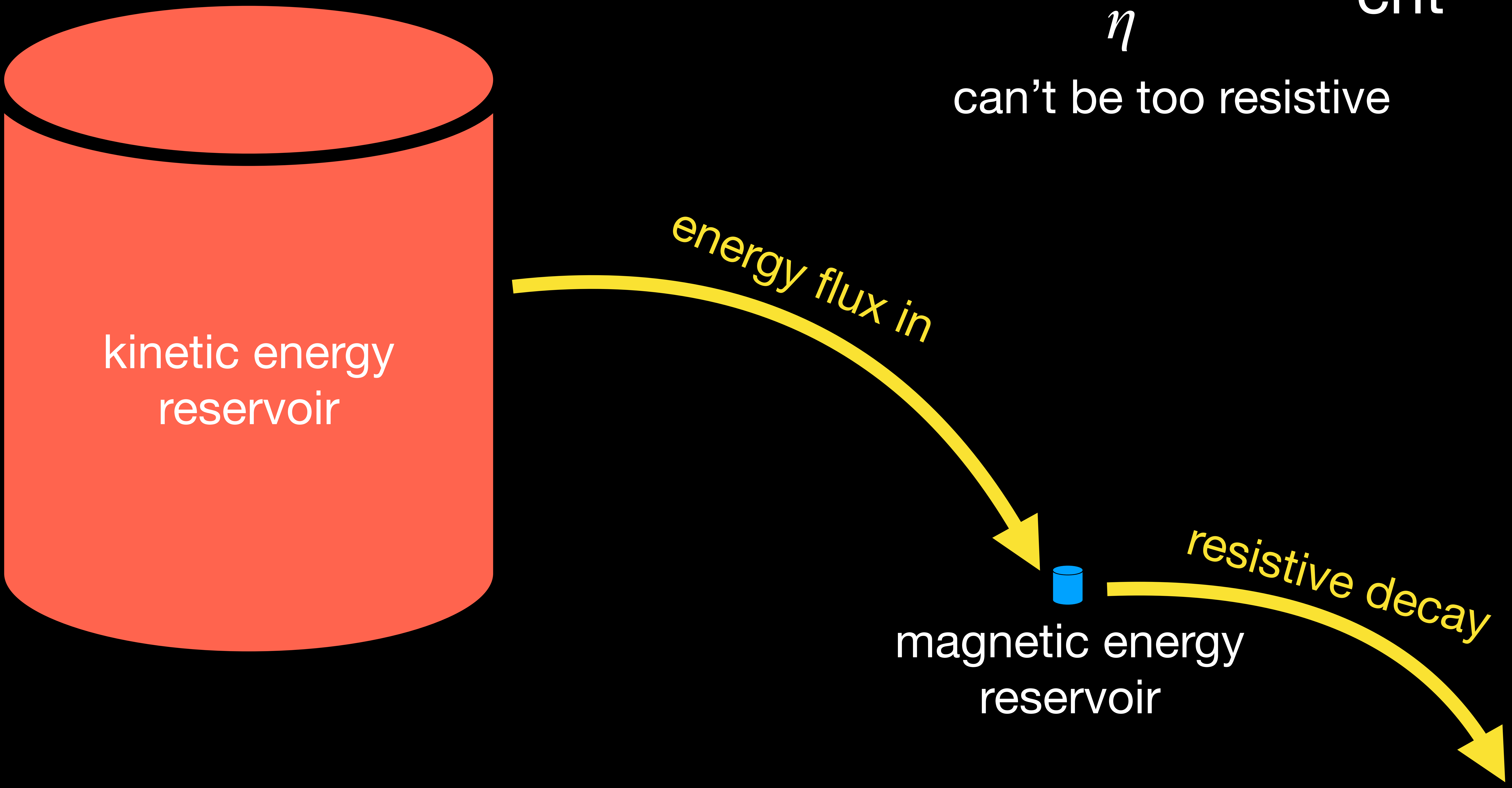
What is a magnetic dynamo?
Starting with a weak seed magnetic field



What is a magnetic dynamo?
Growth

$$\mathrm{Rm} \sim \frac{U_0 L}{\eta} > \mathrm{Rm}_{\mathrm{crit}}$$

can't be too resistive



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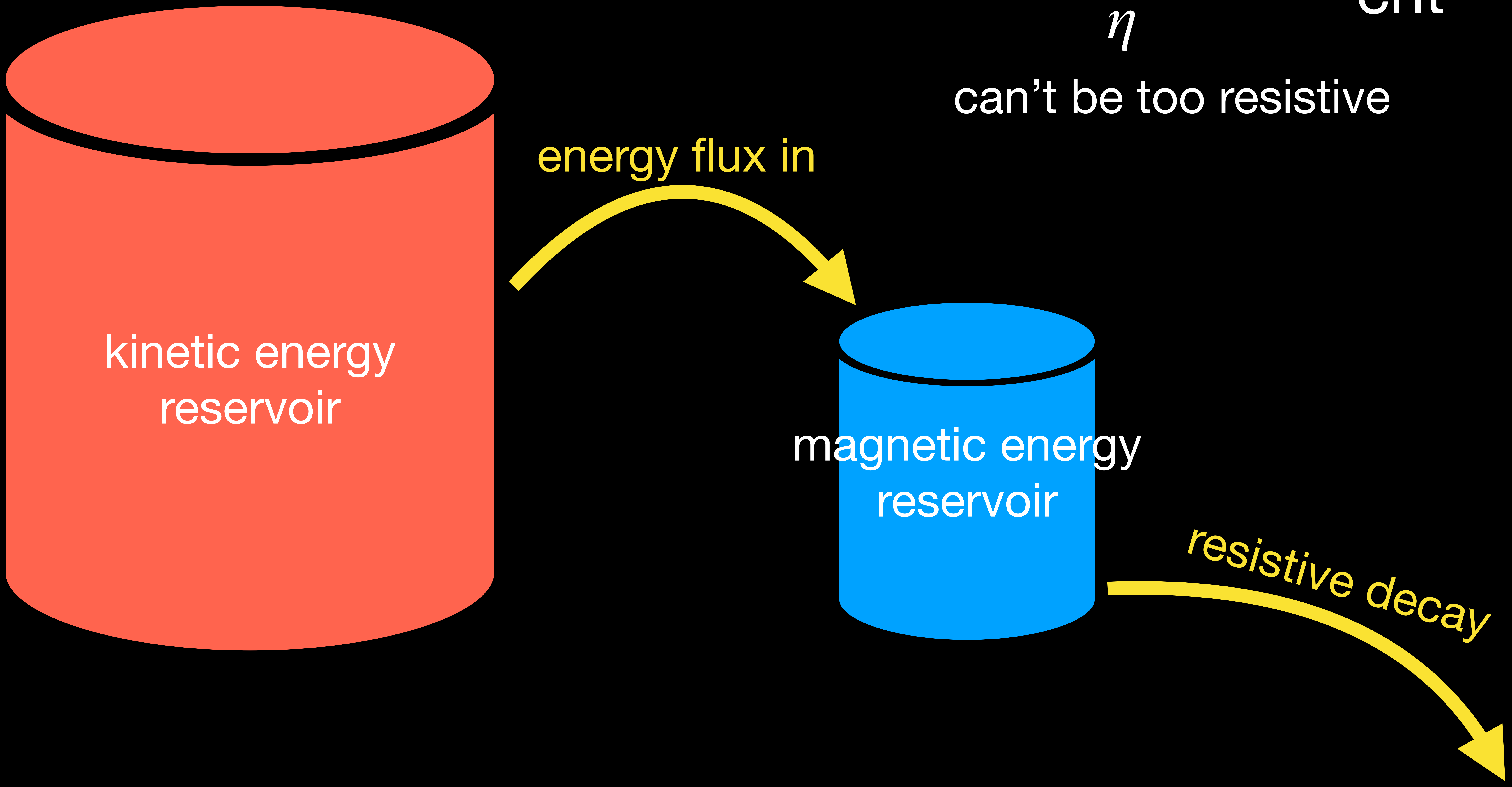
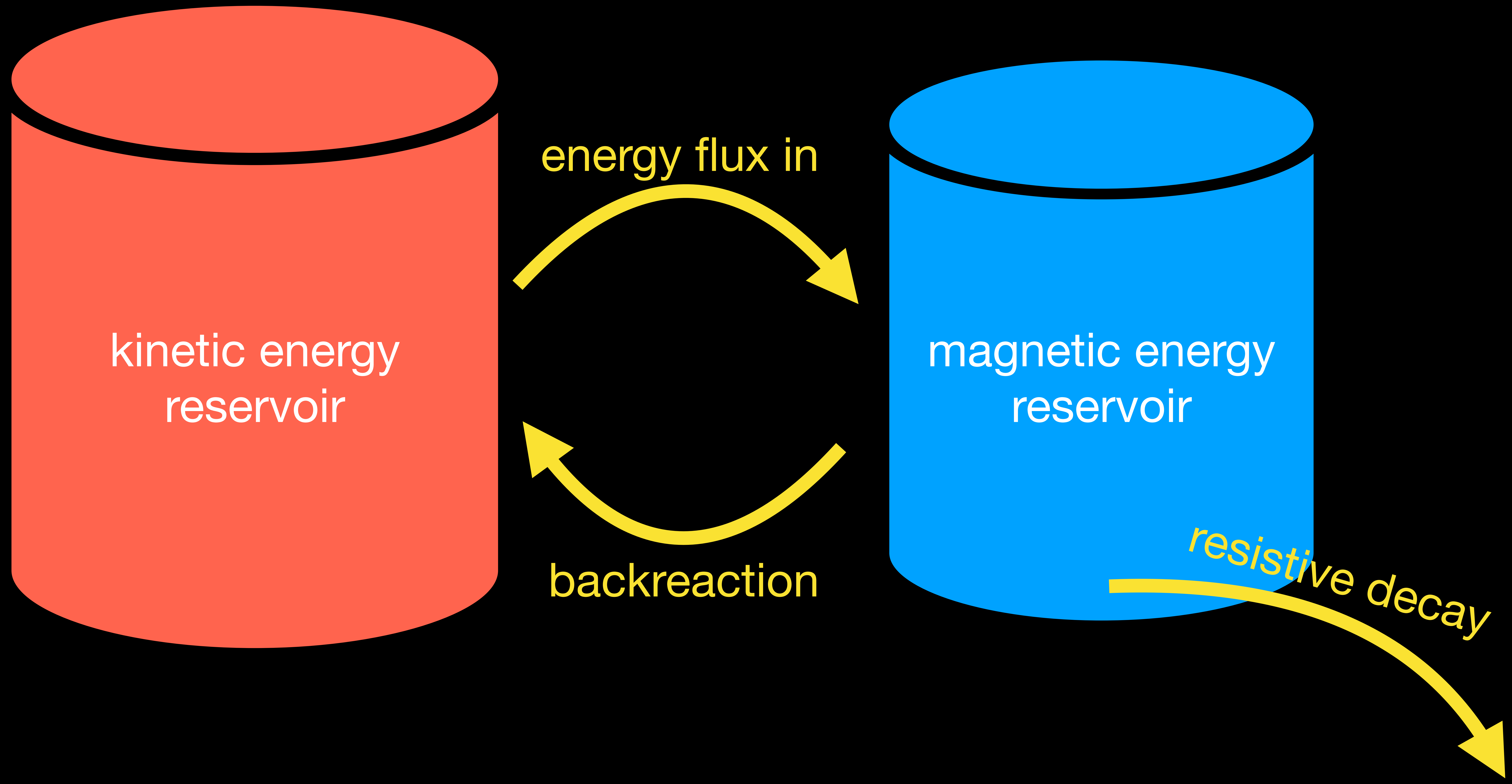
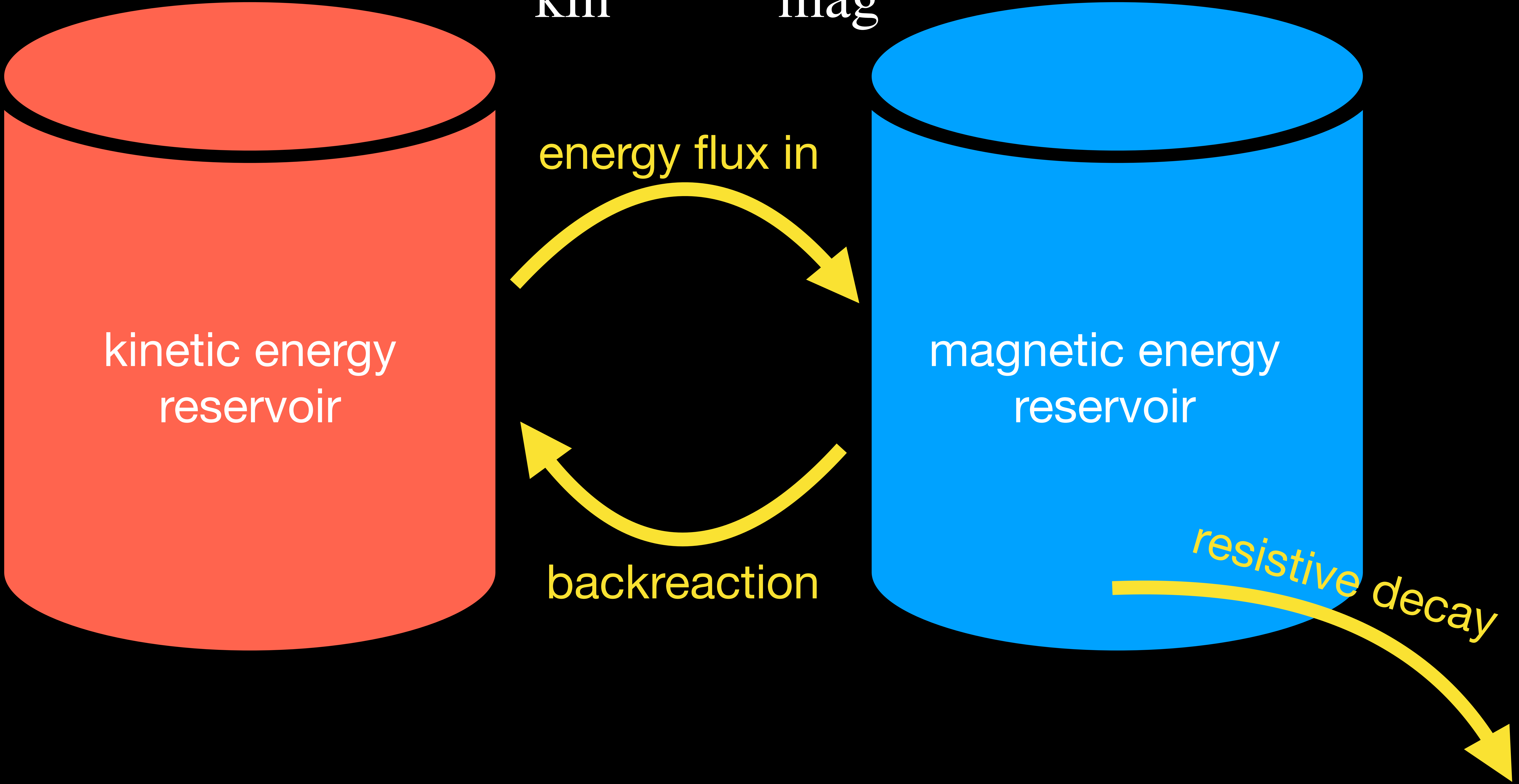


What is a magnetic dynamo?
Nonlinearities and backreaction



What is a magnetic dynamo?
Saturation

$$\mathcal{E}_{\rm kin} \sim \mathcal{E}_{\rm mag}$$

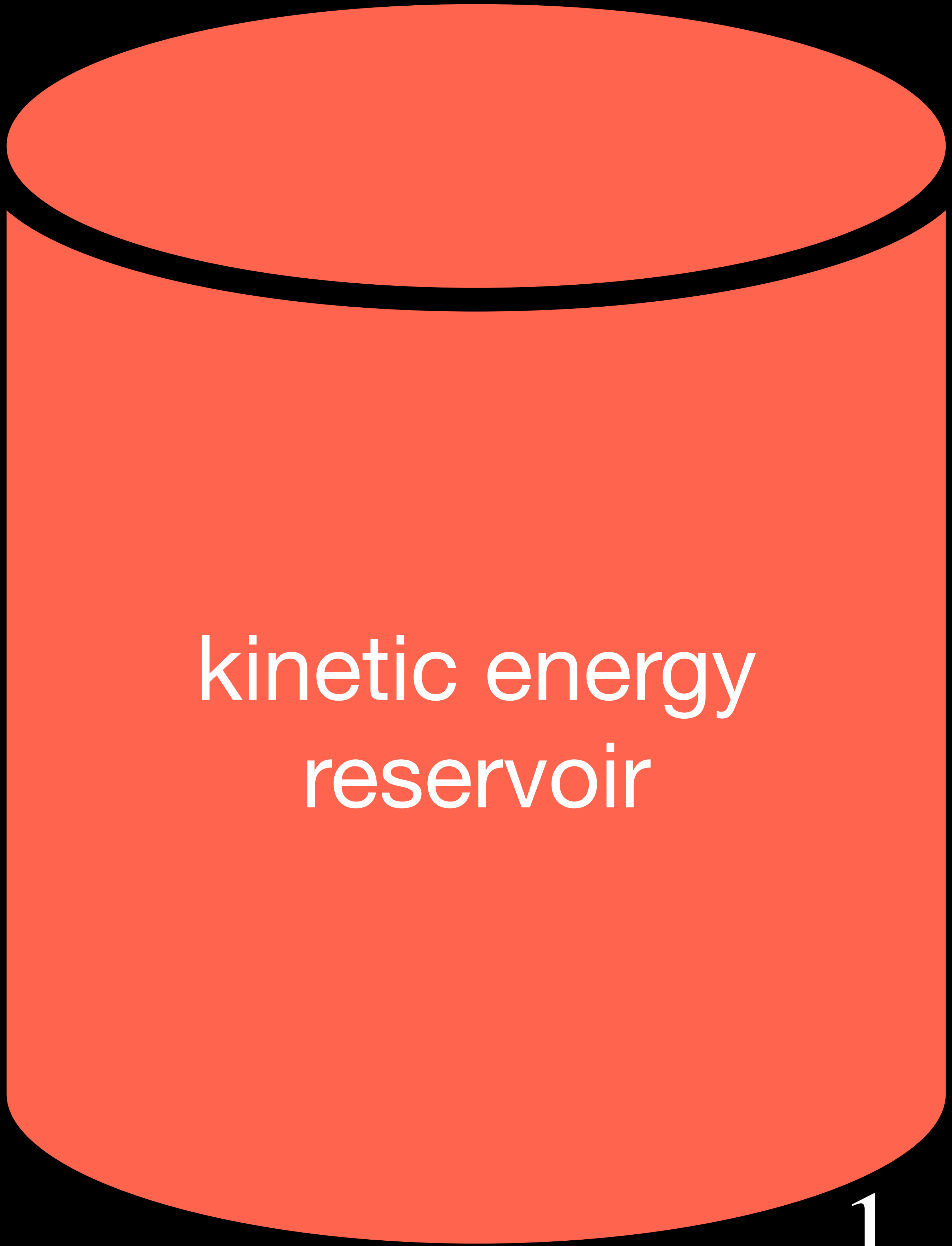


Again more quantitative: What is a magnetic dynamo?

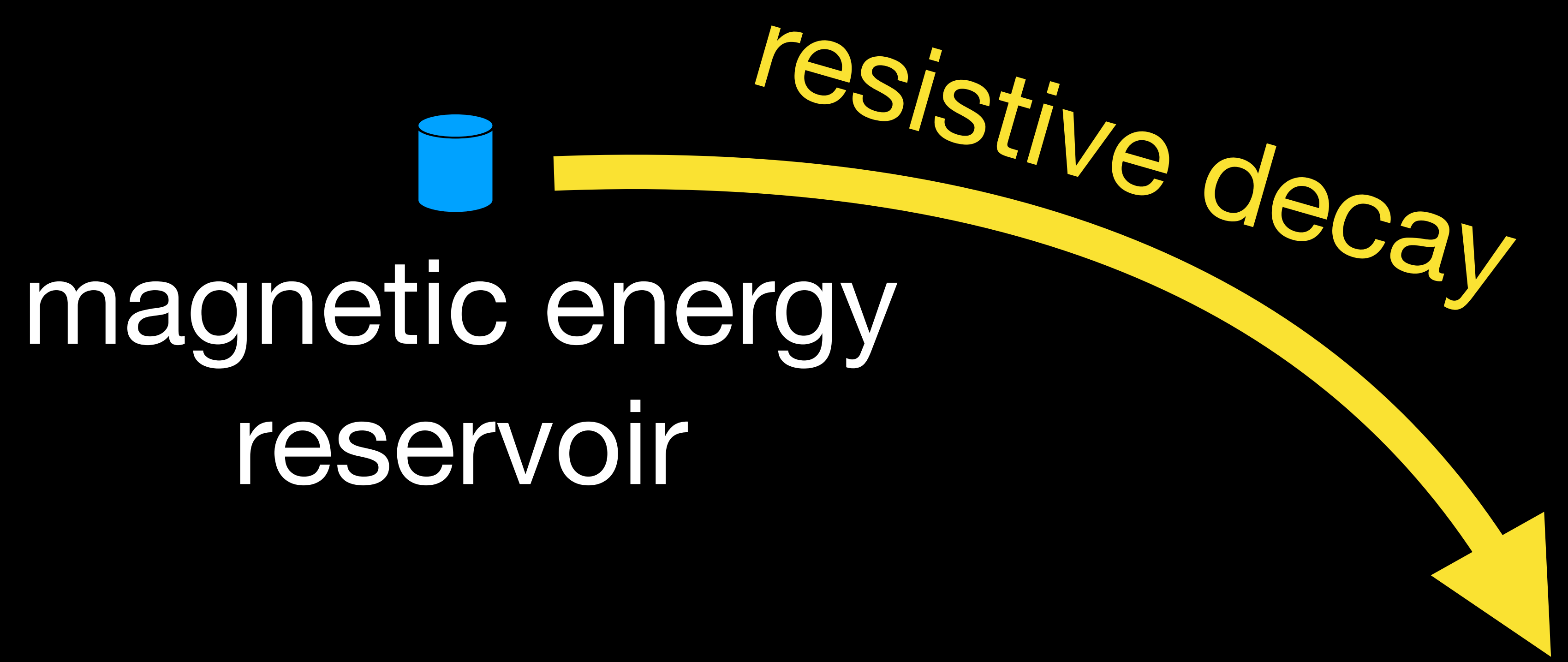
Starting with a weak seed magnetic field

$$\mathrm{Rm} \sim \frac{u_0 \ell_0}{\eta}$$

$$\mathrm{Re} \sim \frac{u_0 \ell_0}{\nu}$$



$$\mathbf{b} \cdot \partial_t \mathbf{b} = \partial_t \mathscr{E}_{\mathrm{mag}} = \frac{1}{\mathrm{Rm}} \mathbf{b} \cdot \mathbb{D}_\eta(\mathbf{b})$$



$$\left\langle \mathbf{u} \cdot \nabla \cdot \mathbb{F}_{\mathbf{u}} + \mathbf{u} \cdot \mathbf{f}_{\mathrm{turb}} \right\rangle_t = \frac{1}{\mathrm{Re}} \left\langle \mathbf{u} \cdot \mathbb{D}_\nu(\mathbf{u}) \right\rangle_t$$

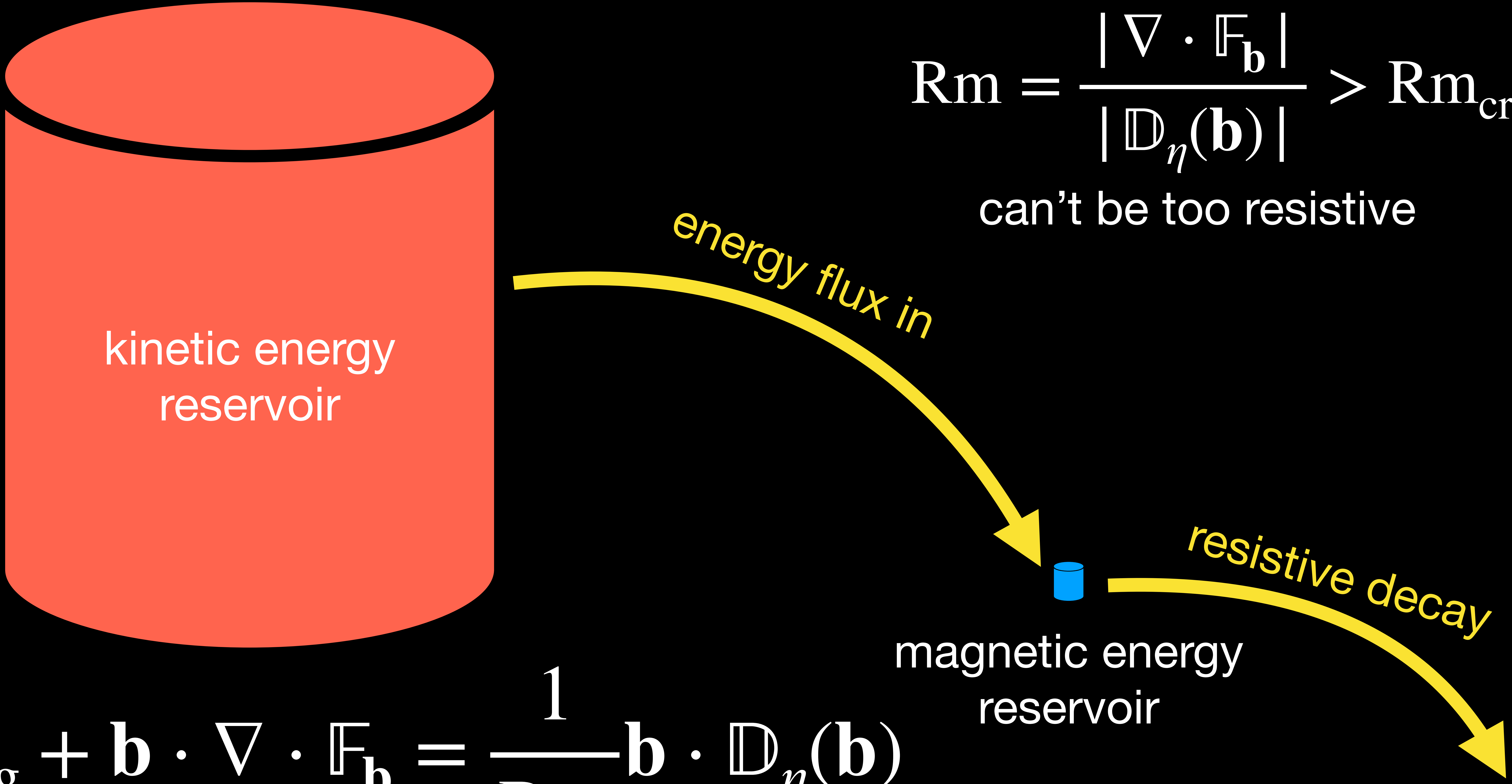
$$\varepsilon_{\mathrm{in}} = \varepsilon_{\mathrm{out}}$$

Again more quantitative: What is a magnetic dynamo?

Growth

$$\mathrm{Rm} = \frac{|\nabla \cdot \mathbb{F}_{\mathbf{b}}|}{|\mathbb{D}_{\eta}(\mathbf{b})|} > \mathrm{Rm}_{\mathrm{crit}}$$

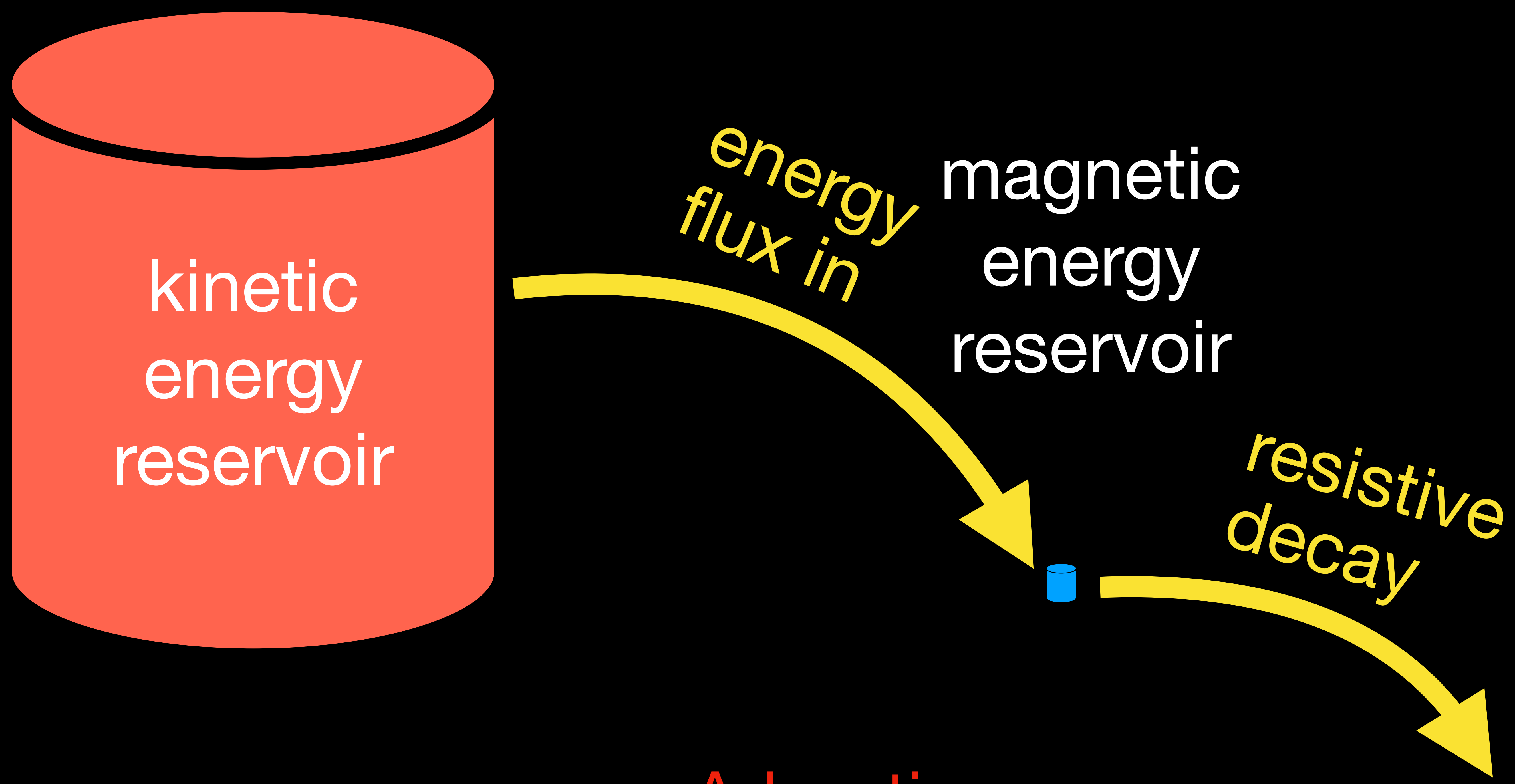
can't be too resistive



$$\partial_t \mathscr{E}_{\mathrm{mag}} + \mathbf{b} \cdot \nabla \cdot \mathbb{F}_{\mathbf{b}} = \frac{1}{\mathrm{Rm}} \mathbf{b} \cdot \mathbb{D}_{\eta}(\mathbf{b})$$

Again more quantitative: What is a magnetic dynamo?

Flux terms



$$\mathrm{Rm} = \frac{|\nabla \cdot \mathbb{F}_{\mathbf{b}}|}{|\mathbb{D}_\eta(\mathbf{b})|} > \mathrm{Rm}_{\mathrm{crit}}$$

can't be too resistive

$$\mathbf{u} \otimes \mathbf{u} : \nabla \otimes \mathbf{u} = u_j u_i \partial_i u_j$$

Advection

compression

$$\mathbf{b} \cdot \nabla \cdot \mathbb{F}_{\mathbf{b}} = \mathbf{b} \otimes \mathbf{u} : \nabla \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{b} : \nabla \otimes \mathbf{u} - \frac{1}{2}\mathbf{b} \otimes \mathbf{b} : (\nabla \cdot \mathbf{u})\mathbb{I}$$

coupling to velocity gradients

Again more quantitative: What is a magnetic dynamo?

Gradient coupling

$$\mathbf{b}\cdot\nabla\cdot\mathbb{F}_{\mathbf{b}} = \underbrace{\mathbf{b}\otimes\mathbf{u}:\nabla\otimes\mathbf{b}}_{\text{advection}} - \underbrace{\mathbf{b}\otimes\mathbf{b}:\nabla\otimes\mathbf{u}}_{\text{coupling to velocity gradients}} + \frac{1}{2}\underbrace{\mathbf{b}\otimes\mathbf{b}:(\nabla\cdot\mathbf{u})\mathbb{I}}_{\text{compression}}$$

$$\nabla\otimes\mathbf{u} = \mathbb{A} + \mathbb{S} + \mathbb{B}$$

$$\mathbb{S} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) - \frac{1}{3}\delta_{ij}\partial_k u_k$$

volume preserving

$$\mathbb{A}_{ij} = \frac{1}{2}\left(\partial_i u_j - \partial_j u_i\right) = -\frac{1}{2}\epsilon_{ijk}\omega_k$$

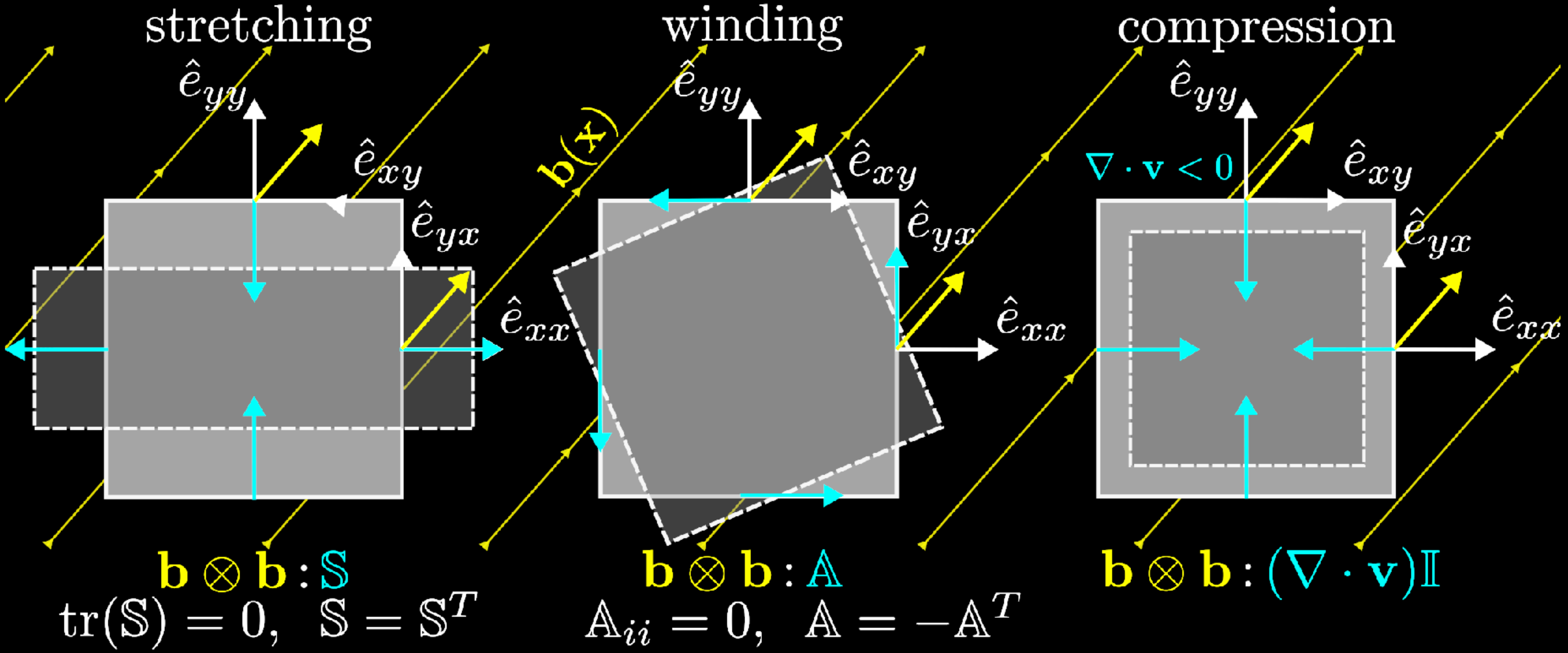
volume preserving

$$\mathbb{B}_{ij} = \frac{1}{3}\delta_{ij}\partial_k u_k$$

volume changing

Again more quantitative: What is a magnetic dynamo?

Gradient tensor decomp.



Beattie + (incl. Bhattcharjee 2024; https://arxiv.org/abs/2312.03984)

Again more quantitative: What is a magnetic dynamo?

Gradient tensor decomp.

$$\mathbf{b} \cdot \nabla \cdot \mathbb{F}_{\mathbf{b}} = \overbrace{\mathbf{b} \otimes \mathbf{u} : \nabla \otimes \mathbf{b}}^{\text{advection}}$$

$$-\underbrace{\mathbf{b} \otimes \mathbf{b} : \mathbb{S}(\mathbf{u})}_{\text{stretching}} - \underbrace{\mathbf{b} \otimes \mathbf{b} : \mathbb{A}(\mathbf{u})}_{\text{rotation}} + \underbrace{\frac{1}{6}\mathbf{b} \otimes \mathbf{b} : (\nabla \cdot \mathbf{u})\mathbb{I}}_{\text{compression}}$$

Again more quantitative: What is a magnetic dynamo?

Gradient tensor decomp.

$$\mathbf{b}\cdot\nabla\cdot\mathbb{F}_{\mathbf{b}} = \mathbf{b}\otimes\mathbf{u} : \nabla\otimes\mathbf{b}$$

$$-\mathbf{b}\otimes\mathbf{b} : \mathbb{S}(\mathbf{u}) - \mathbf{b}\otimes\mathbf{b} : \mathbb{A}(\mathbf{u}) + \frac{1}{6}\mathbf{b}\otimes\mathbf{b} : (\nabla\cdot\mathbf{u})\mathbb{I}$$

rotation ($\mathbb{A}$ is actually a representation of $\mathfrak{SO}(3)$)

symmetric antisymmetric

$$\mathbf{b}\otimes\mathbf{b} : \mathbb{A} = 0$$

Always exactly orthogonal! <u>You can never grow magnetic field flux with rotation operator!</u>

Again more quantitative: What is a magnetic dynamo?
Energy flux

Remaining terms

$$\mathbf{b}\cdot\nabla\cdot\mathbb{F}_{\mathbf{b}} = \mathbf{b}\otimes\mathbf{u} : \nabla\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{b} : \mathbb{S}(\mathbf{u}) + \frac{1}{6}\mathbf{b}\otimes\mathbf{b} : (\nabla\cdot\mathbf{u})\mathbb{I}$$

Each term could potentially describe an interaction between three difference modes (triad interactions)…

e.g., $\mathbf{b}(\mathbf{k}'), \quad \mathbf{b}(\mathbf{k}''), \quad \mathbf{b}(\mathbf{k}'''), \quad \mathbf{u}(\mathbf{k}'), \quad \dots$

$$[\mathbf{b}\cdot\nabla\cdot\mathbb{F}_{\mathbf{b}}] \sim U^3/L \quad \text{energy flux density}$$

Again more quantitative: What is a magnetic dynamo?

Cascade versus dynamo

Momentum conservation:

doner, receiver, mediator

$$\mathbf{k}' + \mathbf{k}'' + \mathbf{k}''' = 0$$

or

antisymmetry property:
(giveth = - taketh)

doner, receiver, mediator

$$\mathbf{k}' \xrightarrow{\mathbf{k}''} \mathbf{k}''' = -\,\mathbf{k}''' \xrightarrow{\mathbf{k}''} \mathbf{k}'$$

Can extract these interactions directly from stochastic magnetic fields by constructing filtered vector fields

$$\mathbf{b}' = \mathbf{b}(\mathbf{r}') = \int \delta^3(\mathbf{k} - \mathbf{k}')\mathbf{b}(\mathbf{k})\exp\{2\pi i\mathbf{k}\cdot\mathbf{r}\}$$

Again more quantitative: What is a magnetic dynamo?

Cascade versus dynamo

$$\mathbf{k}' \xrightarrow{\mathbf{k}''} \mathbf{k}'''$$

Rewrite magnetic energy equation in terms of triad interactions:

$$\mathbf{b}''' \cdot \partial_t \mathbf{b}' = \partial_t \mathscr{E}_{\text{mag}} = -\,\mathbf{b}''' \cdot \nabla \cdot \mathbb{F}_{\mathbf{b}'} + \frac{1}{\text{Rm}} \mathbf{b}''' \cdot \mathbb{D}_\eta(\mathbf{b}')$$

where

$$\mathbf{b}''' \cdot \nabla \cdot \mathbb{F}_{\mathbf{b}'} = \mathbf{b}''' \otimes \mathbf{u}'' : \nabla \otimes \mathbf{b}'$$

$$-\mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}' + \frac{1}{2} \mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}'')\mathbb{I}$$

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$$= \mathbf{b}''' \otimes \mathbf{u}'' : \nabla \otimes \mathbf{b}' - \mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}' + \frac{1}{2}\mathbf{b}' \otimes \mathbf{b}''' : (\nabla \cdot \mathbf{u}'')\mathbb{I}$$

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$$- \underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}'}_{\text{kinetic to magnetic energy transfer}}$$

looks like flux generation via compression… it's not

Again more quantitative: What is a magnetic dynamo?

Growth

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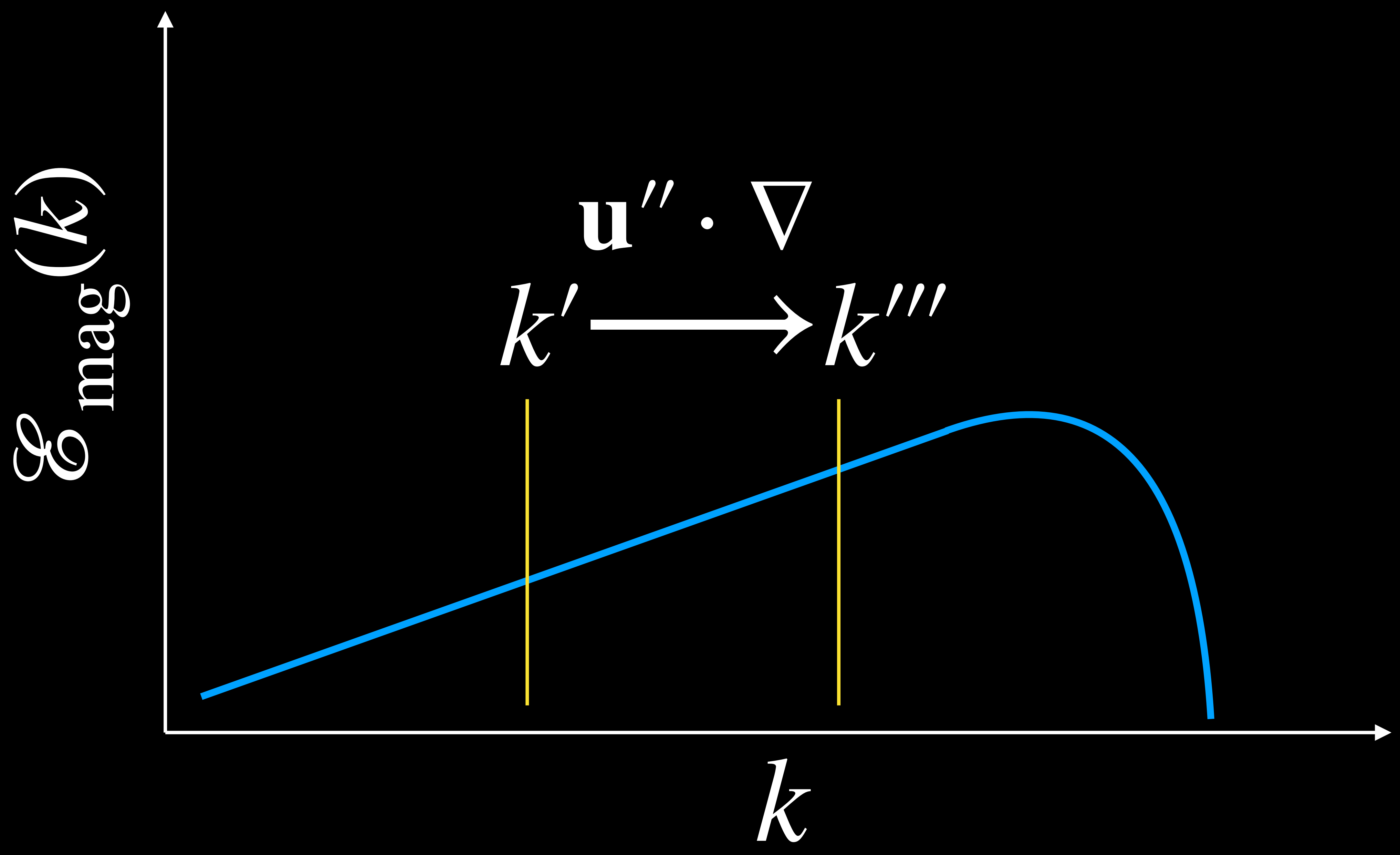

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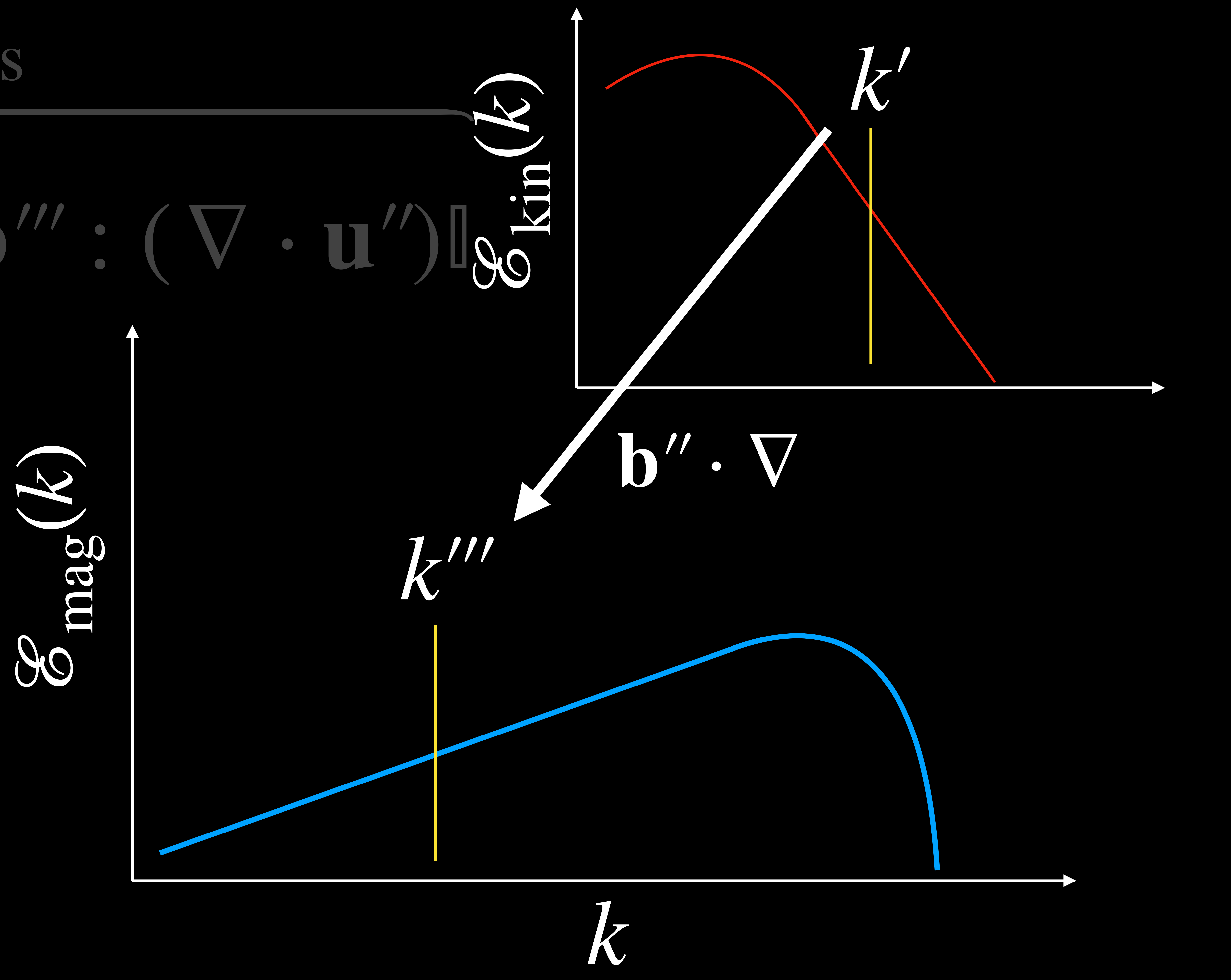

Again more quantitative: What is a magnetic dynamo?

Dynamo and compression

$$\mathbf{k}' \xrightarrow{\mathbf{k}''} \mathbf{k}'''$$

Kinetic to magnetic energy flux

$$\mathbf{b}''' \otimes \mathbf{b}'' : \nabla \otimes \mathbf{u}' = \overbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{S}(\mathbf{u}')}^{\text{dynamo}} + \underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{B}(\mathbf{u}')}_{\text{flux compression}}$$

antisymmetric property

$$\overbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{S}(u')}^{\text{dynamo}} = -\mathbf{u}''' \otimes \mathbf{b}'' : \mathbb{S}(b') = -\,\mathbf{u}''' \otimes \underbrace{\mathbf{b}'' : \nabla \otimes \mathbf{b}'}_{\text{magnetic tension}}$$

$$\underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{B}(u')}_{\text{flux compression}} = -\frac{1}{2}\mathbf{u}''' \cdot \underbrace{\nabla \otimes (\mathbf{b}' \cdot \mathbf{b}'')}_{\text{magnetic pressure}}$$

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We learn:

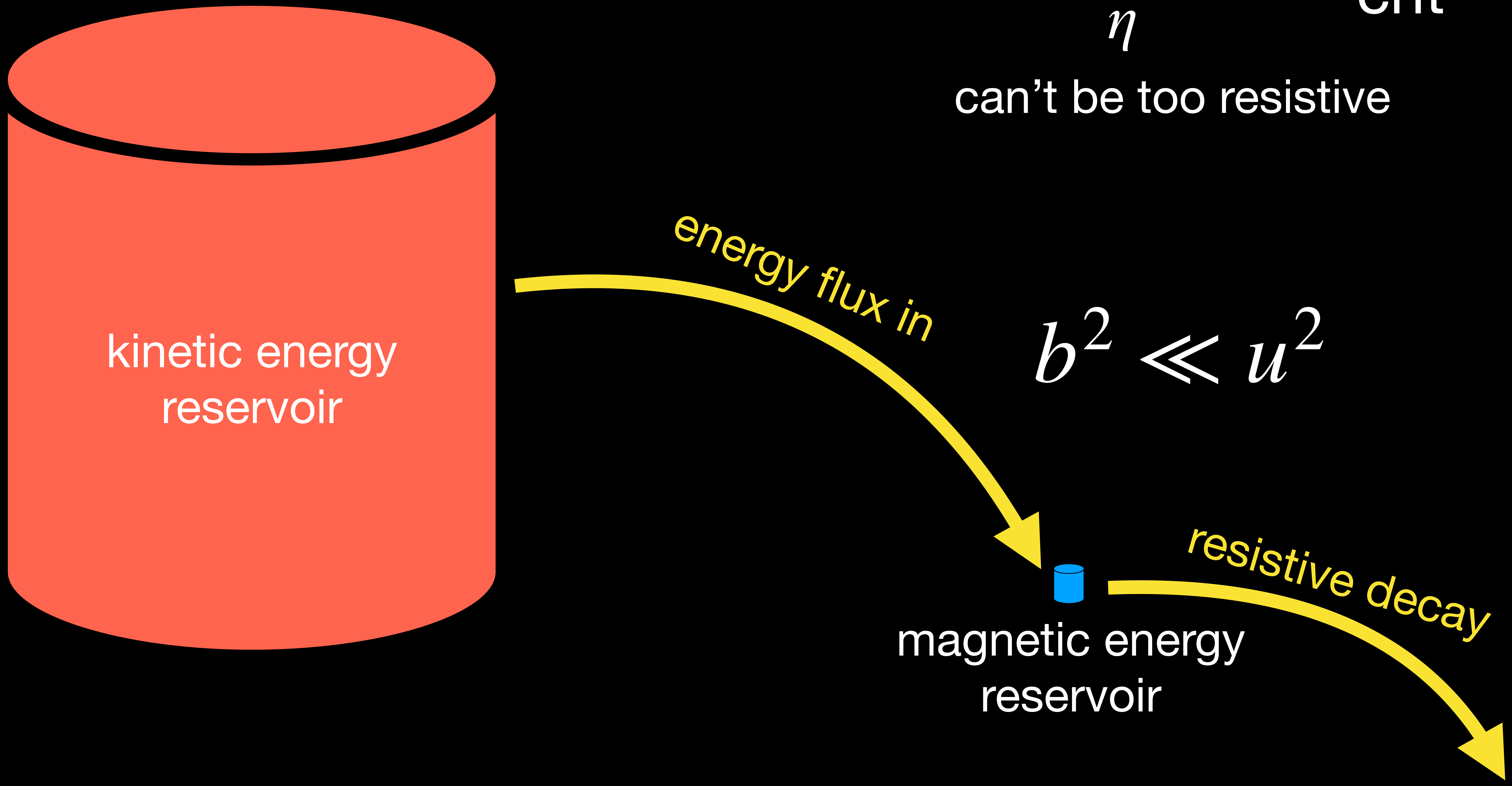
- which term gives rise to dynamo (cascade, flux compression)
- tension **always** balances with stretching from dynamo,
- and pressure **always** balances flux compression + flux freezing

Again more quantitative: What is a magnetic dynamo?

Fast growth stage

$$\mathrm{Rm} \sim \frac{U_0 L}{\eta} > \mathrm{Rm}_{\mathrm{crit}}$$

can't be too resistive



$$b^2 \ll u^2$$

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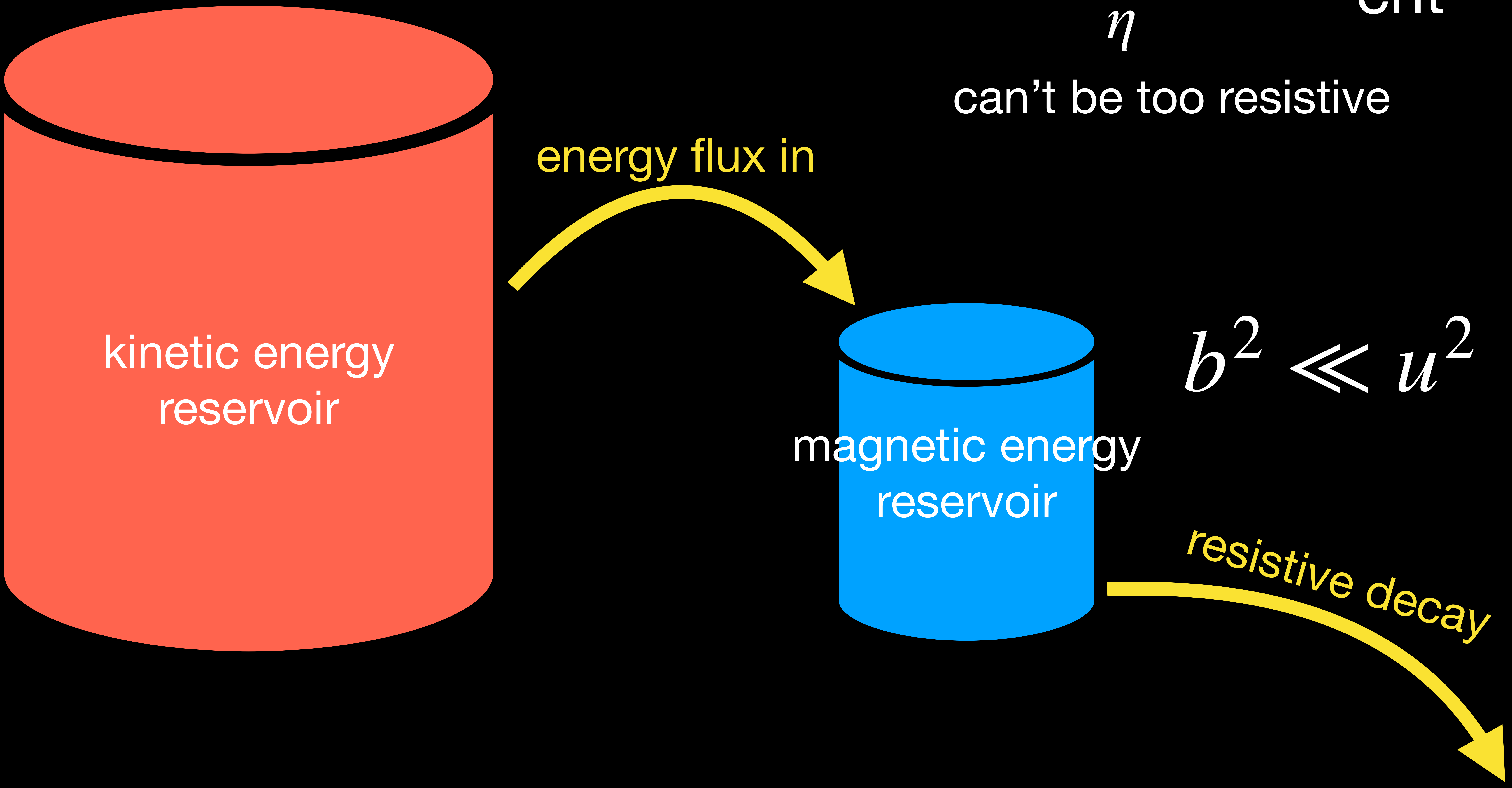

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$$\underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{B}(u')}_{\text{flux compression}} = -\frac{1}{2}\mathbf{u}''' \cdot \underbrace{\boxed{\nabla \otimes (\mathbf{b}' \cdot \mathbf{b}'')}}_{\text{magnetic pressure}}$$

- $\mathbf{b}$ coupled to $\mathbf{u}$ via $\sim b^2/\ell$ in momentum equation
- For $b^2 \ll u^2$, $\mathbf{u}$ only a very weak function of $\mathbf{b}$

No $\mathbf{u}(\mathbf{b})$ back reaction!

independent of $\mathbf{b}$

$$\partial_t \mathbf{b} + \nabla \cdot (\mathbf{u} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{u}) = \frac{1}{\mathrm{Rm}} \mathbb{D}_\eta(\mathbf{b})$$

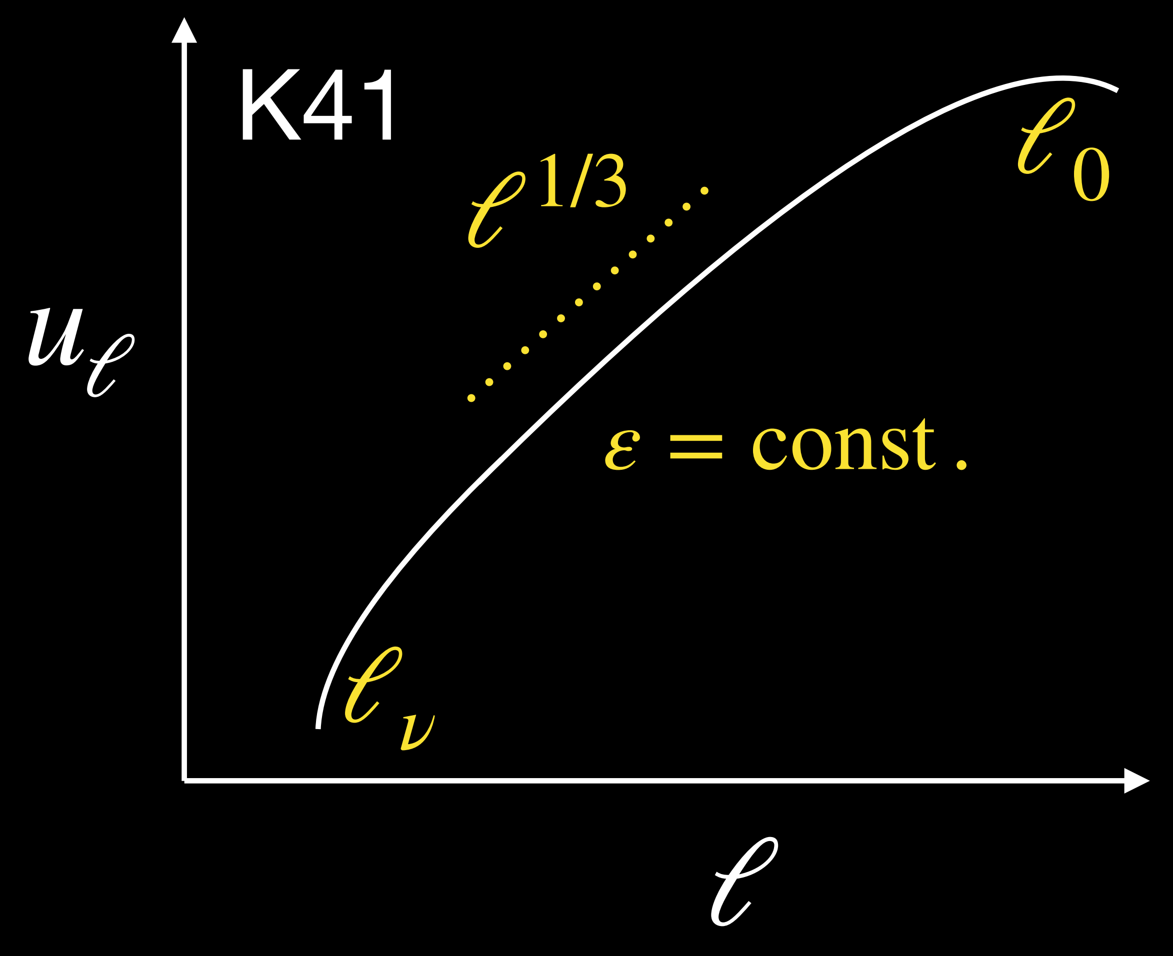
linear in $\mathbf{b}$

Again more quantitative: What is a magnetic dynamo?

Fast growth stage — integral energy

Rewrite magnetic energy eq. and take 1^{st} moments

$$\mathrm{d}_t \left\langle \mathscr{E}_{\text{mag}} \right\rangle = 2\left(\gamma - \eta k_{\text{rms}}^2\right) \left\langle \mathscr{E}_{\text{mag}} \right\rangle$$



where

$$\gamma = \left\langle \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} : \nabla \otimes \mathbf{u} \right\rangle, \, k_{\text{rms}}^2 = \left\langle \nabla \otimes \hat{\mathbf{b}} : \nabla \otimes \hat{\mathbf{b}} \right\rangle$$

Growth sourced via $\nabla \otimes \mathbf{u}$… which comes from the (hydro) turbulence

$$\ell_\nu \sim \mathrm{Re}^{-3/4} \ell_0, \qquad u_\ell \sim (\varepsilon \ell)^{1/3}, \qquad \varepsilon \sim u^3/\ell$$

cascade — velocity scaling — constant energy flux

*assuming homogenous, isotropic, incompressible Kolmogorov-type turbulence

Again more quantitative: What is a magnetic dynamo?

Fast growth stage — dynamo engine is the viscous scale

Growth sourced via $\nabla \otimes \mathbf{u}$… which comes from the (hydro) turbulence

$$u_\ell/\ell \sim \varepsilon^{1/3}\ell^{-2/3}, \qquad t_\ell \sim \ell^{2/3}.$$

velocity gradients strongest at small scales

Dynamical timescales smallest at small scales

hence growth rate dominated by the smallest possible scales of the flow gradients

$$\gamma = \left\langle \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} : \nabla \otimes \mathbf{u} \right\rangle \sim u_\nu/\ell_\nu \sim 1/t_\nu$$

i.e., the viscous eddy scale ℓ_ν (on $\ell < \ell_\nu$ flow is diffusive).

Again more quantitative: What is a magnetic dynamo?

Fast growth stage — integral energy

hence growth rate dominated by the smallest possible scales of the flow gradients

$$\gamma = \left\langle \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} : \nabla \otimes \mathbf{u} \right\rangle \sim u_\nu / \ell_\nu \sim 1/t_\nu,$$

put in units of outer scale turnover time $t_0 = \ell_0 / u_0$

$$t_0 \gamma \sim t_0 / t_\nu, \qquad t_0 / t_\nu \sim (\ell_0 / \ell_\nu)^{2/3} \sim \left(\mathrm{Re}^{3/4}\right)^{2/3} \sim \mathrm{Re}^{1/2},$$

and to summarise,

$$\gamma = \left\langle \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} : \nabla \otimes \mathbf{u} \right\rangle \sim 1/t_\nu \sim \mathrm{Re}^{1/2} / t_0 \, .$$

Again more quantitative: What is a magnetic dynamo?

Fast growth stage — integral energy

$$\gamma = \left\langle \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} : \nabla \otimes \mathbf{u} \right\rangle \sim 1/t_\nu \sim \mathrm{Re}^{1/2}/t_0 \,.$$

Consider the cold neutral medium in ISM. $T = 80\,\mathrm{K}$.

$$\mathrm{Re} \sim 4 \times 10^{10}, \qquad t_0 \sim 3\,\mathrm{Myr}, \qquad t_0/\mathrm{Re}^{1/2} \sim 10\,\mathrm{years} \,.$$

Every 10 years the cold phase of the Galaxy can increase its field by a factor of $e \approx 3$.
This is a diffusion-free turbulent dynamo because we assumed

$$\mathrm{d}_t \langle \mathscr{E}_{\mathrm{mag}} \rangle = 2\gamma_{\mathrm{eff}} \langle \mathscr{E}_{\mathrm{mag}} \rangle, \quad \gamma_{\mathrm{eff}} = \gamma - \eta k_{\mathrm{rms}}^2, \quad \eta k_{\mathrm{rms}}^2 \longrightarrow 0$$

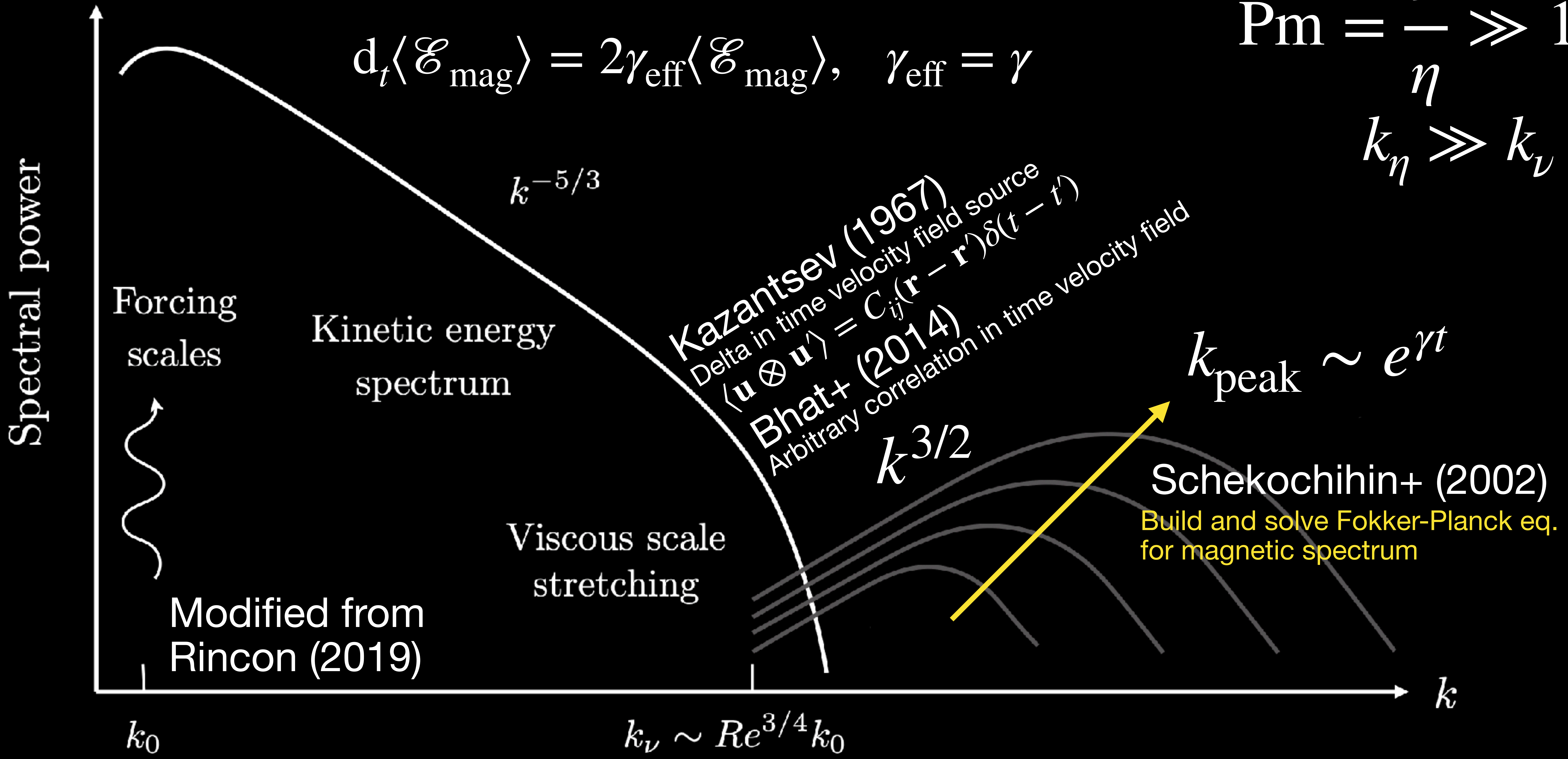
Again more quantitative: What is a magnetic dynamo?

First growth stage: diffusion-free regime

$$\mathrm{d}_t \langle \mathscr{E}_{\mathrm{mag}} \rangle = 2\gamma_{\mathrm{eff}} \langle \mathscr{E}_{\mathrm{mag}} \rangle, \quad \gamma_{\mathrm{eff}} = \gamma$$

$$\mathrm{Pm} = \frac{\nu}{\eta} \gg 1$$

$$k_\eta \gg k_\nu$$



Again more quantitative: What is a magnetic dynamo?

First growth stage: diffusion-free regime

$$d_t\langle \mathscr{E}_{\rm mag}\rangle = 2\gamma_{\rm eff}\langle \mathscr{E}_{\rm mag}\rangle, \quad \gamma_{\rm eff} = \gamma$$

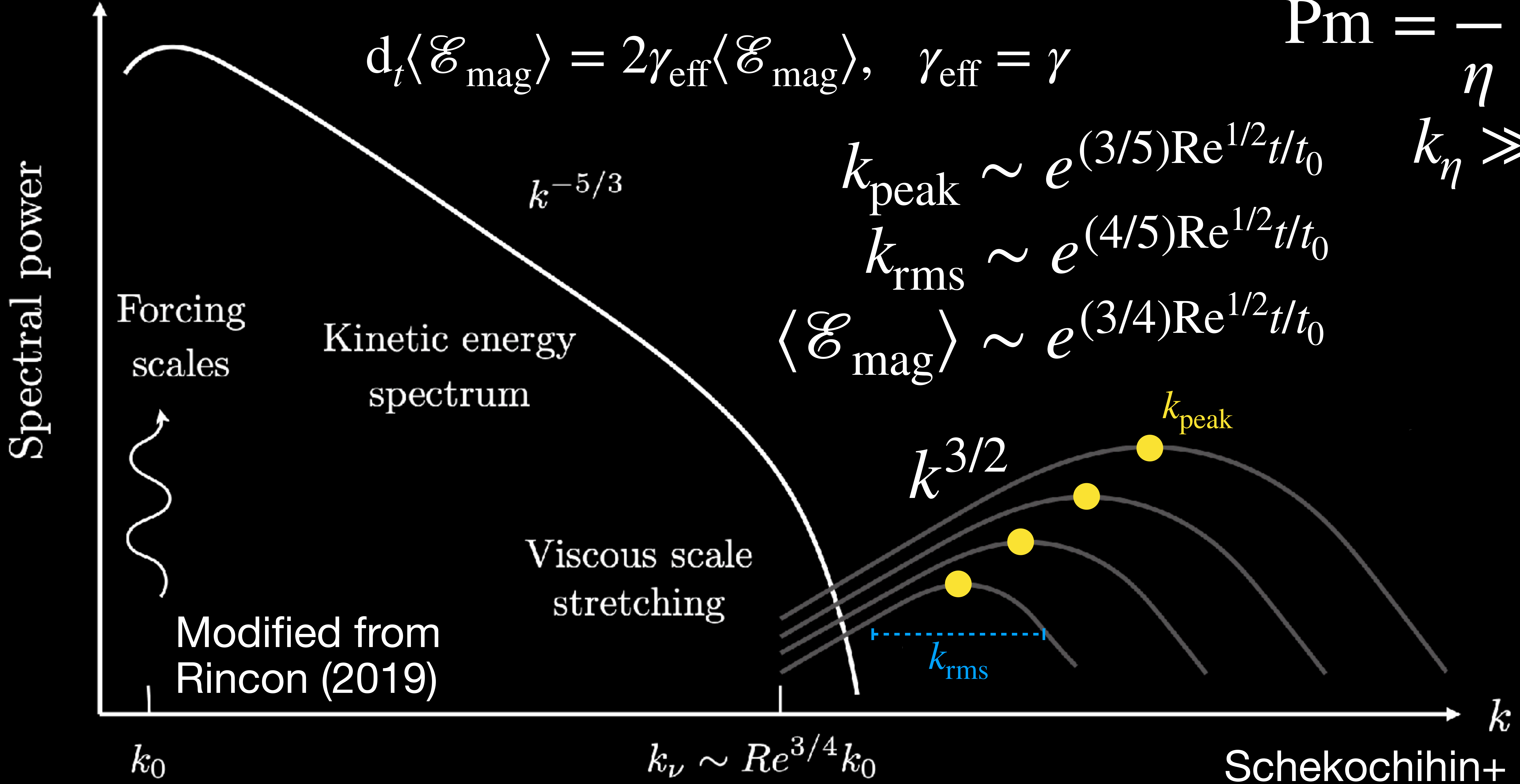
$$\mathrm{Pm} = \frac{\nu}{\eta} \gg 1$$

$$k_\eta \gg k_\nu$$

$$k_{\rm peak} \sim e^{(3/5)\mathrm{Re}^{1/2} t/t_0}$$

$$k_{\rm rms} \sim e^{(4/5)\mathrm{Re}^{1/2} t/t_0}$$

$$\langle \mathscr{E}_{\rm mag}\rangle \sim e^{(3/4)\mathrm{Re}^{1/2} t/t_0}$$



Schekochihin+ (2002)

Again more quantitative: What is a magnetic dynamo?

Second growth stage: kinematic regime

$$\mathrm{Pm} = \frac{\nu}{\eta} \gg 1$$

$$k_\eta \gg k_\nu$$

stretching at the viscous scale

$$\frac{u_\nu}{\ell_\nu} \sim \frac{\eta}{\ell_\eta^2}$$

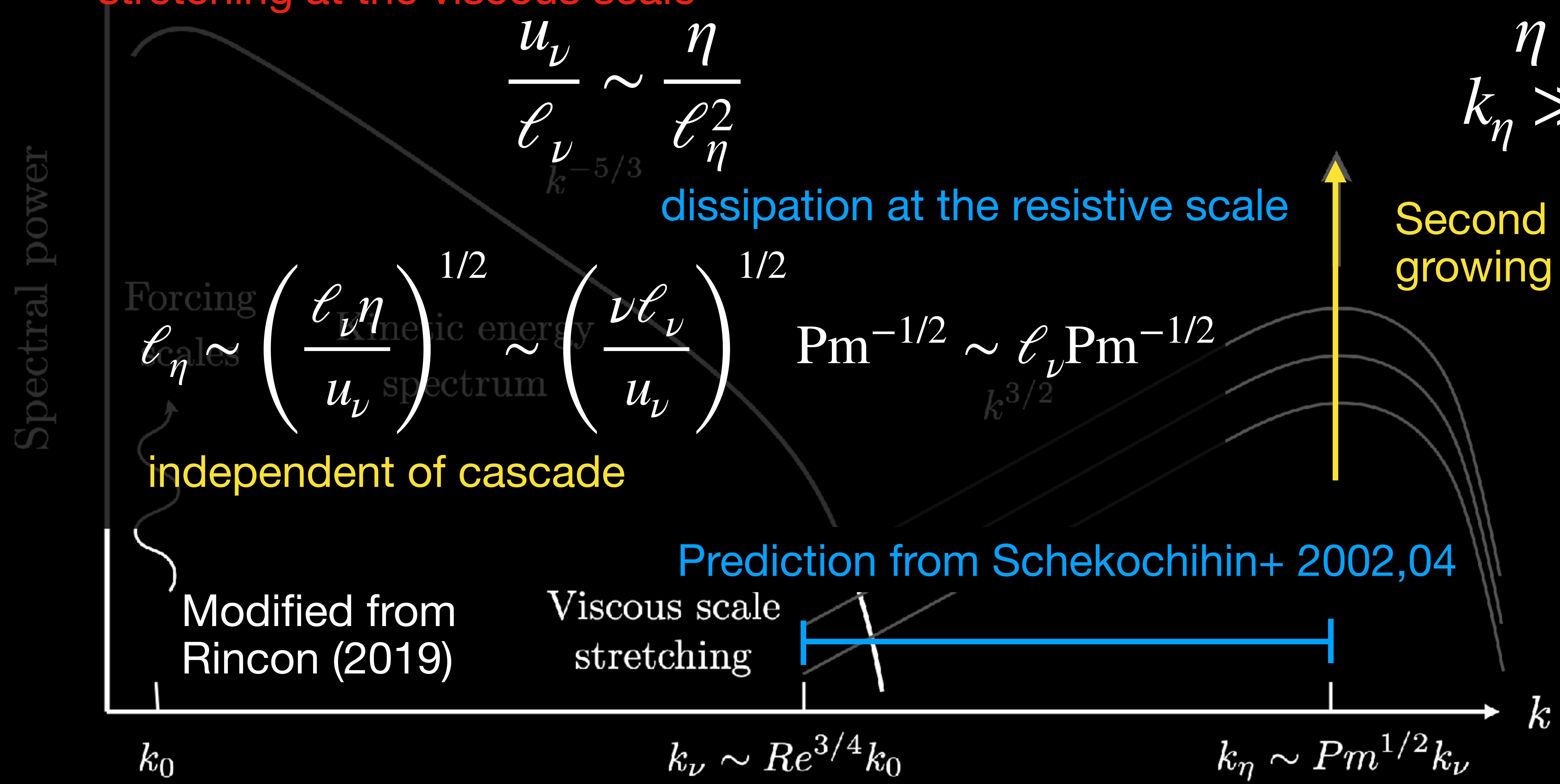
dissipation at the resistive scale

Second fastest growing stage

$$\ell_\eta \sim \left(\frac{\ell_\nu \eta}{u_\nu}\right)^{1/2} \sim \left(\frac{\nu \ell_\nu}{u_\nu}\right)^{1/2} \mathrm{Pm}^{-1/2} \sim \ell_\nu \mathrm{Pm}^{-1/2}$$

independent of cascade

Prediction from Schekochihin+ 2002,04



Modified from Rincon (2019)

Again more quantitative: What is a magnetic dynamo?

Second growth stage: kinematic regime

$$\mathrm{Pm} = \frac{\nu}{\eta} \gg 1$$

$$k_\eta \gg k_\nu$$

$$\mathrm{d}_t \langle \mathscr{E}_{\mathrm{mag}} \rangle = 2\gamma_{\mathrm{eff}} \langle \mathscr{E}_{\mathrm{mag}} \rangle, \quad \gamma_{\mathrm{eff}} = \gamma - \eta k_{\mathrm{rms}}^2,$$

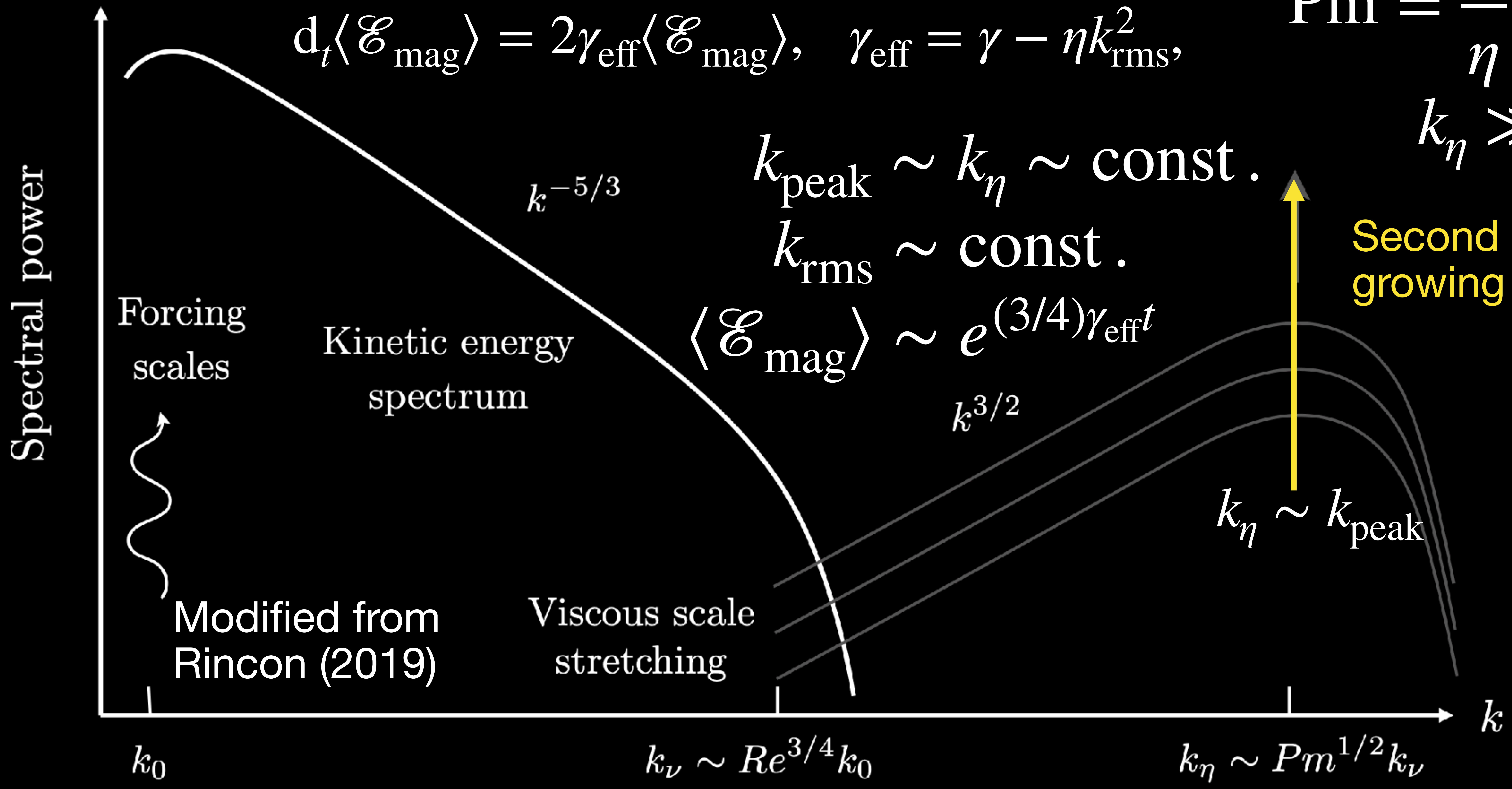
$$k_{\mathrm{peak}} \sim k_\eta \sim \mathrm{const.}$$

$$k_{\mathrm{rms}} \sim \mathrm{const.}$$

$$\langle \mathscr{E}_{\mathrm{mag}} \rangle \sim e^{(3/4)\gamma_{\mathrm{eff}} t}$$

Second fastest growing stage

$$k_\eta \sim k_{\mathrm{peak}}$$

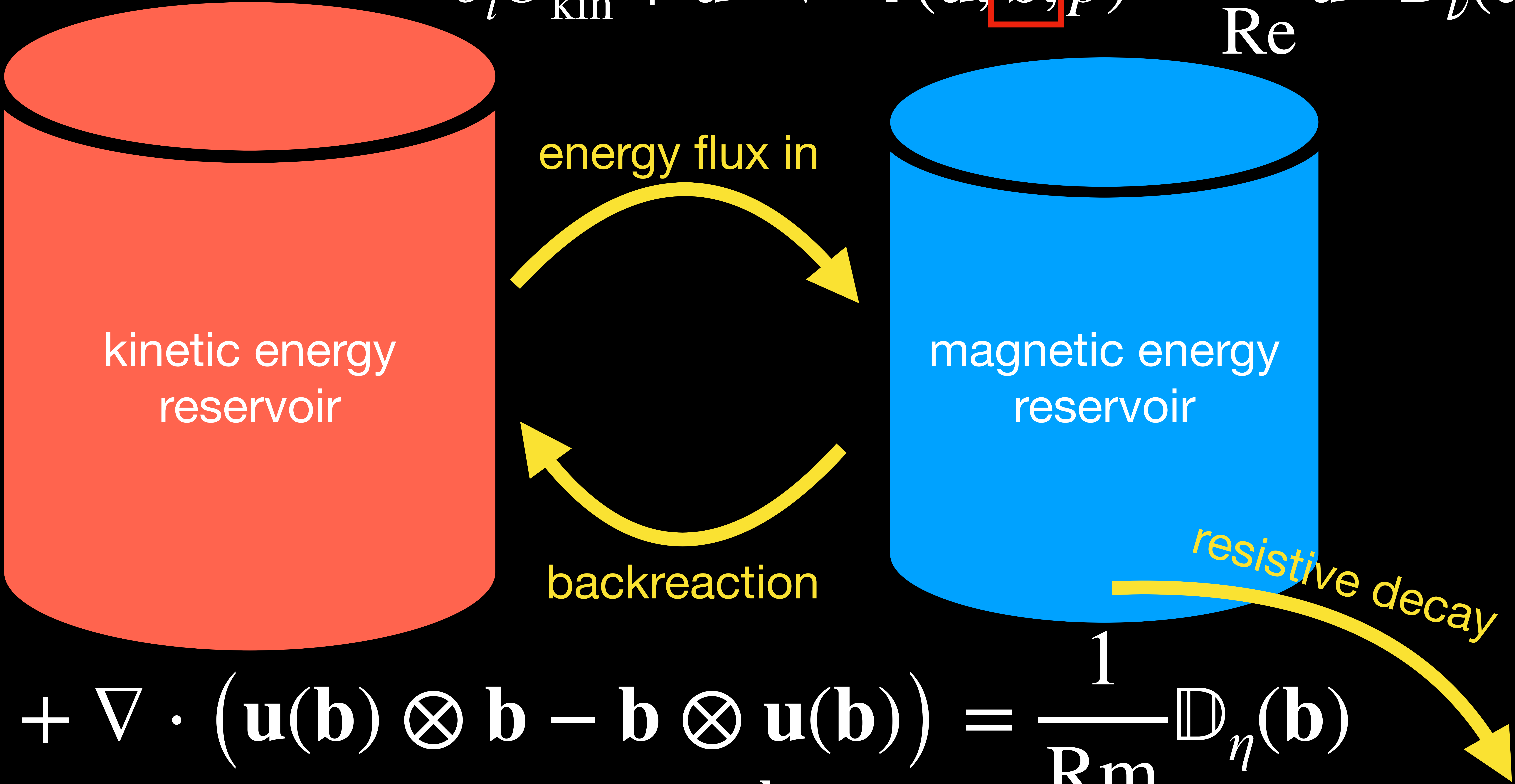


Modified from Rincon (2019)

Again more quantitative: What is a magnetic dynamo?

Linear growth and backreaction

$$\partial_t \mathscr{E}_{\mathrm{kin}} + \mathbf{u} \cdot \nabla \cdot \mathbb{F}(\mathbf{u}, \boxed{\mathbf{b},} p) = \frac{1}{\mathrm{Re}} \mathbf{u} \cdot \mathbb{D}_\nu(\mathbf{u})$$



$$\partial_t \mathbf{b} + \nabla \cdot \Big(\mathbf{u}(\mathbf{b}) \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{u}(\mathbf{b}) \Big) = \frac{1}{\mathrm{Rm}} \mathbb{D}_\eta(\mathbf{b})$$

induction equation now nonlinear in $\mathbf{b}$

Again more quantitative: What is a magnetic dynamo?
Linear growth and backreaction

The backreaction is directly encoded in the antisymmetric property

$$\overbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{S}(u')}^{\text{dynamo}} = -\mathbf{u}''' \otimes \mathbf{b}'' : \mathbb{S}(b') = -\,\mathbf{u}''' \otimes \underbrace{\mathbf{b}'' : \nabla \otimes \mathbf{b}'}_{\text{magnetic tension}}$$

$$\underbrace{\mathbf{b}''' \otimes \mathbf{b}'' : \mathbb{B}(u')}_{\text{flux compression}} = -\frac{1}{2}\mathbf{u}''' \cdot \underbrace{\nabla \otimes (\mathbf{b}' \cdot \mathbf{b}'')}_{\text{magnetic pressure}}$$

$\mathscr{T}_{ub}(k', k'', k''')$

Energy flux from kinetic to magnetic energy densities

$-\mathscr{T}_{bu}(k', k'', k''')$

Energy flux from magnetic to kinetic energy densities

Always present, always balancing $\mathscr{T}_{ub}$, but now non-negligible.

Again more quantitative: What is a magnetic dynamo?

Linear growth and backreaction

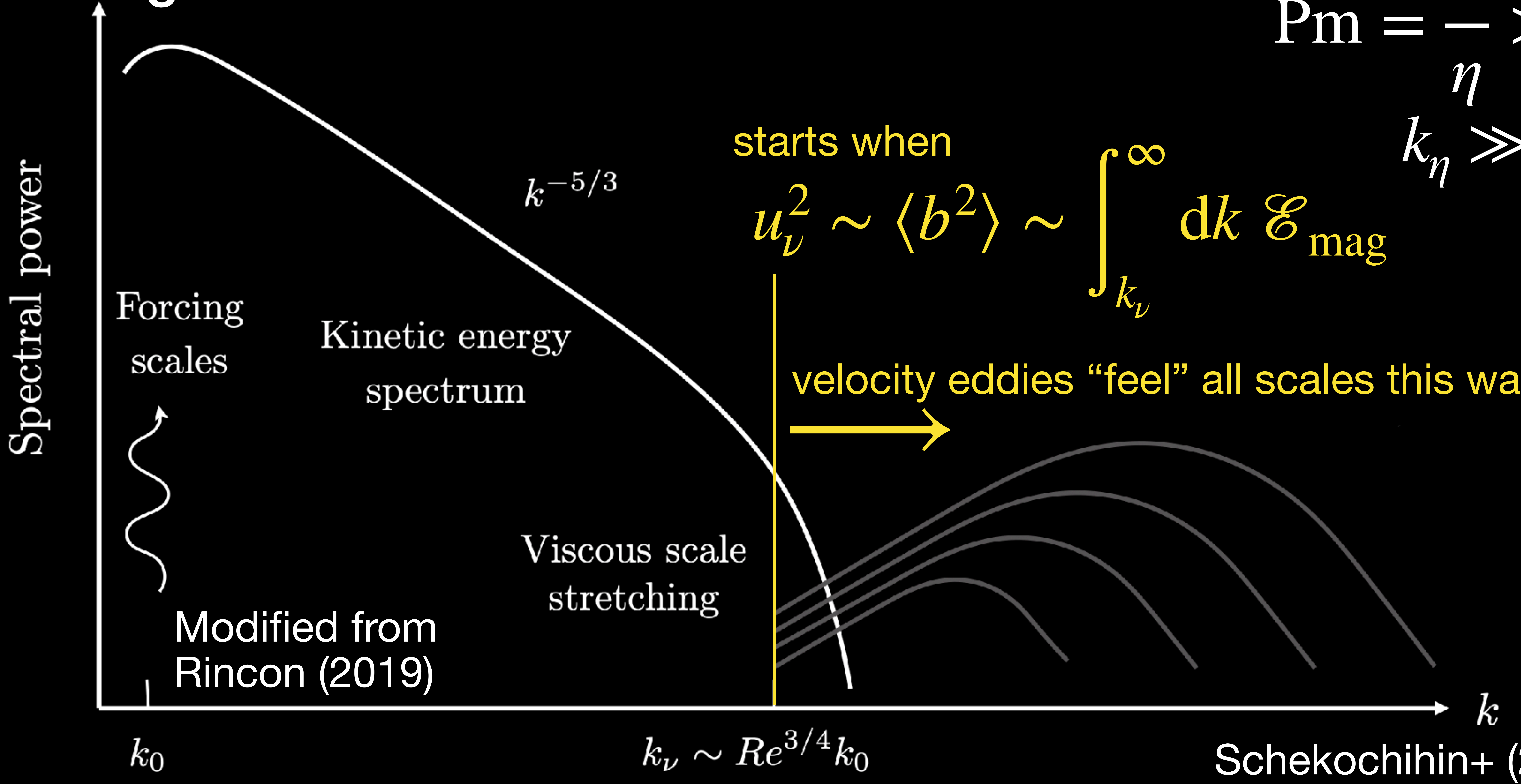
$$\mathrm{Pm} = \frac{\nu}{\eta} \gg 1$$

$$k_\eta \gg k_\nu$$

starts when

$$u_\nu^2 \sim \langle b^2 \rangle \sim \int_{k_\nu}^{\infty} \mathrm{d}k\, \mathscr{E}_{\mathrm{mag}}$$

velocity eddies "feel" all scales this way



Schekochihin+ (2002)

Again more quantitative: What is a magnetic dynamo?

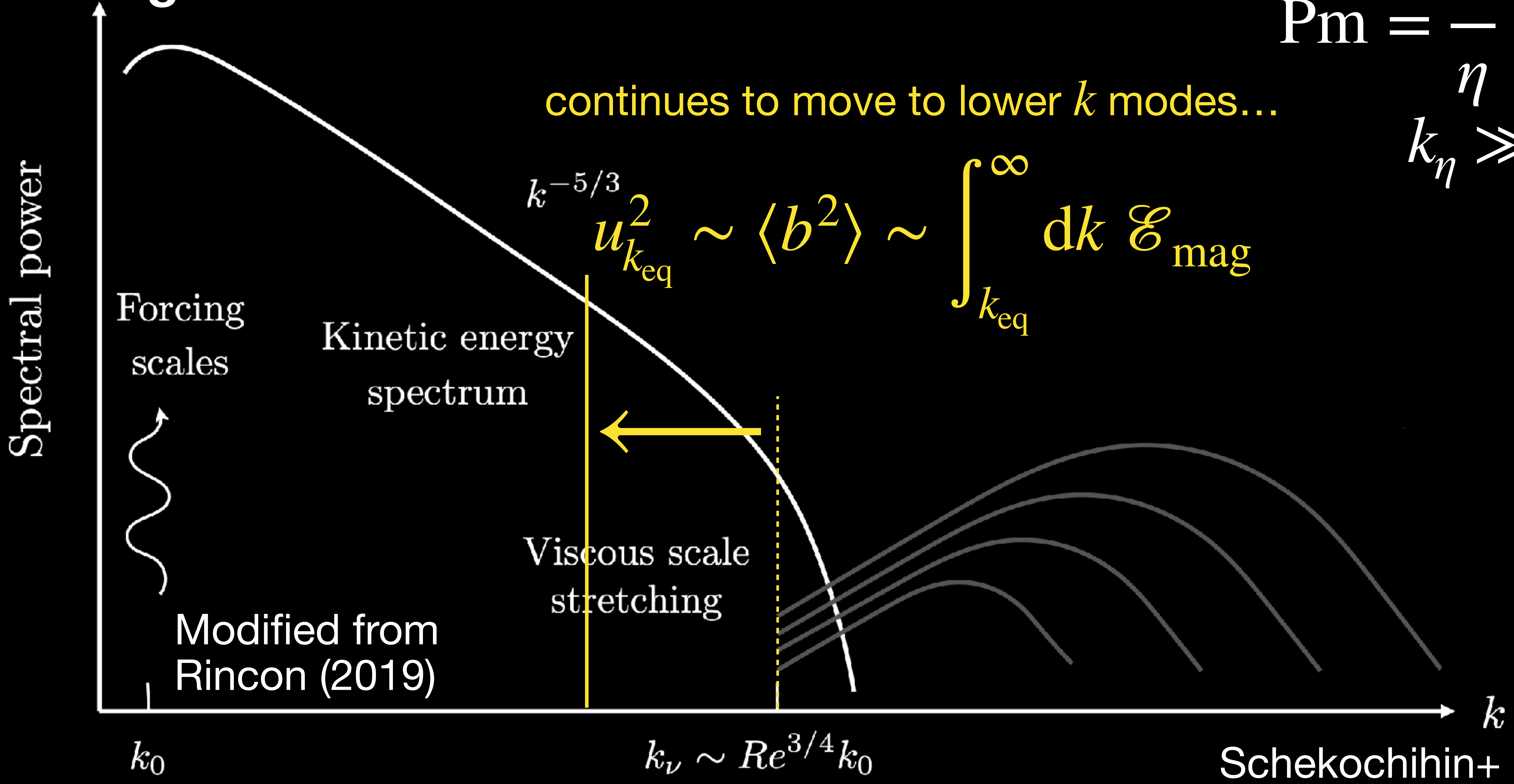
Linear growth and backreaction

$$\mathrm{Pm} = \frac{\nu}{\eta} \gg 1$$

$$k_\eta \gg k_\nu$$

continues to move to lower k modes...

$$u_{k_{\rm eq}}^2 \sim \langle b^2 \rangle \sim \int_{k_{\rm eq}}^{\infty} \mathrm{d}k\, \mathscr{E}_{\rm mag}$$



Modified from Rincon (2019)

Schekochihin+ (2002)

Again more quantitative: What is a magnetic dynamo?
Linear growth and backreaction

$$\mathrm{Pm} = \frac{\nu}{\eta} \gg 1$$

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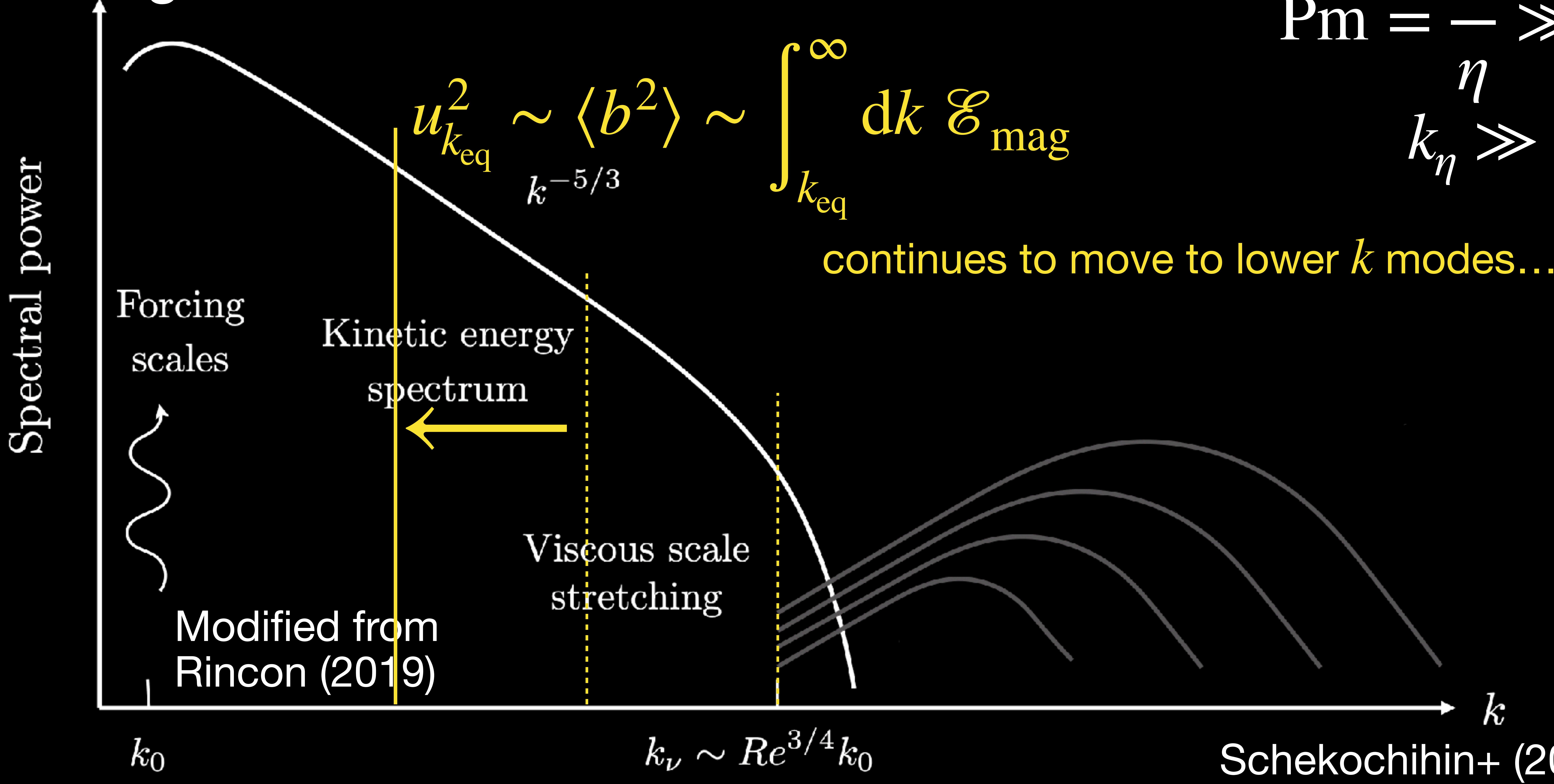


Schekochihin+ (2002)

Again more quantitative: What is a magnetic dynamo?

Linear growth and backreaction

Stretching has to move to larger scales, i.e.,

$$\mathrm{d}_t \langle \mathcal{E}_{\mathrm{mag}} \rangle = 2\gamma_{\mathrm{eff}} \langle \mathcal{E}_{\mathrm{mag}} \rangle, \quad \gamma_{\mathrm{eff}} \sim \frac{u_{\ell_{\mathrm{eq}}}}{\ell_{\mathrm{eq}}},$$

hence,

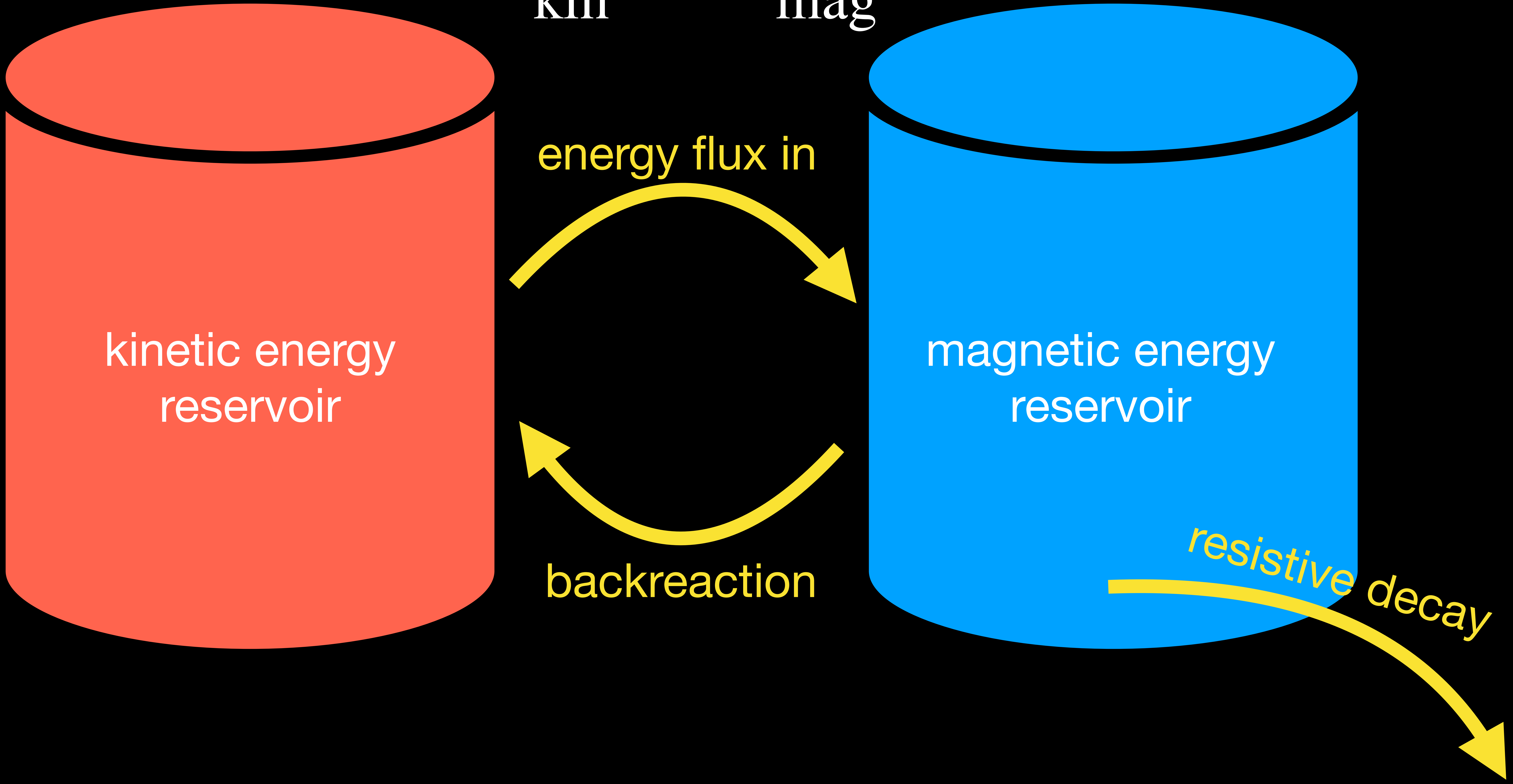
$$\mathrm{d}_t \langle \mathcal{E}_{\mathrm{mag}} \rangle \sim \frac{u_{\ell_{\mathrm{eq}}}}{\ell_{\mathrm{eq}}} \langle \mathcal{E}_{\mathrm{mag}} \rangle \sim \frac{u_{\ell_{\mathrm{eq}}}}{\ell_{\mathrm{eq}}} u_{\ell_{\mathrm{eq}}}^2 \sim \varepsilon_{\mathrm{eq}} \sim \varepsilon,$$

Using simply $\langle \mathcal{E}_{\mathrm{mag}} \rangle \sim u_{\ell_{\mathrm{eq}}}^2$, by definition, and ε is constant everywhere in the cascade

$$\langle \mathcal{E}_{\mathrm{mag}} \rangle \sim \varepsilon t\,.$$

Again more quantitative: What is a magnetic dynamo?
Saturation

$$\mathcal{E}_{\mathrm{kin}} \sim \mathcal{E}_{\mathrm{mag}}$$



More soon….