Kinetic to viscous fluid model via first-order Chapman-Enskog expansion of the BGK-Boltzmann equation

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Contents

1 Boltzmann equation and the BGK collision operator

In its general, nonrelativistic form, the kinetic equation in $(\boldsymbol{x}, \boldsymbol{v})$ includes external forces and reads

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f + \frac{\boldsymbol{F}(\boldsymbol{x}, t)}{m} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} f = C[f], \tag{1}$$

where $\boldsymbol{\nabla}_{\boldsymbol{x}}$ and $\boldsymbol{\nabla}_{\boldsymbol{v}}$ denote gradients in real space and velocity space, respectively, and $\boldsymbol{F}$ is the force acting on a particle (e.g. gravity or the Lorentz force in a plasma). Throughout these notes we assume a neutral, single-species gas, each particle with mass m, in an inertial frame with no external forces, $\boldsymbol{F} = \boldsymbol{0}$, so the streaming operator reduces to $\partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f$. We replace the full collision operator by the Bhatnagar-Gross-Krook (BGK) model,

$$C[f] = -\frac{1}{\tau}\left(f - f^{(0)}\right), \tag{2}$$

which assumes that collisions drive the distribution toward a local Maxwellian $f^{(0)}$ on a characteristic relaxation time τ. The BGK operator preserves mass, momentum, and energy while providing the correct tensorial structure of viscous transport. The Boltzmann equation therefore becomes

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f = -\frac{1}{\tau}\left(f - f^{(0)}\right). \tag{3}$$

The local equilibrium distribution $f^{(0)}$ is the Maxwell-Boltzmann distribution

$$f^{(0)}(\boldsymbol{x}, \boldsymbol{v}, t) = n(\boldsymbol{x}, t)\left(\frac{m}{2\pi k_B T(\boldsymbol{x}, t)}\right)^{3/2} \exp\left[-\frac{m|\boldsymbol{v} - \boldsymbol{u}(\boldsymbol{x}, t)|^2}{2k_B T(\boldsymbol{x}, t)}\right], \tag{4}$$

where

$$n(\boldsymbol{x}, t), \qquad \boldsymbol{u}(\boldsymbol{x}, t), \qquad T(\boldsymbol{x}, t) \tag{5}$$

are the number density, bulk flow velocity, and temperature. These fluid fields are defined as velocity moments of f,

$$n \equiv \int f \; \mathrm{d}^3\boldsymbol{v}, \qquad n\boldsymbol{u} \equiv \int \boldsymbol{v} f \; \mathrm{d}^3\boldsymbol{v}, \qquad \frac{3}{2} n k_B T \equiv \int \frac{1}{2} m|\boldsymbol{v} - \boldsymbol{u}|^2 f \; \mathrm{d}^3\boldsymbol{v}. \tag{6}$$

It is convenient to introduce the mass density $\rho \equiv mn$, the peculiar velocity $\boldsymbol{c} \equiv \boldsymbol{v} - \boldsymbol{u}$, and kinetic and internal energy,

$$\frac{1}{2}\rho u^2 \equiv \int \frac{1}{2} m v^2 f \, \mathrm{d}^3\boldsymbol{v}, \qquad \rho e \equiv \int \frac{1}{2} m c^2 f \, \mathrm{d}^3\boldsymbol{v} = \frac{3}{2} n k_B T, \tag{7}$$

for a monatomic gas. We also define the kinetic energy, pressure (stress) tensor and heat flux by

$$P_{ij} \equiv m \int c_i c_j f \; \mathrm{d}^3\boldsymbol{v}, \qquad q_i \equiv \int \frac{1}{2} m c^2 c_i \, f \; \mathrm{d}^3\boldsymbol{v}. \tag{8}$$

For a local Maxwellian, $P_{ij} = p\,\delta_{ij}$ and $q_i = 0$ with $p = nk_BT$, which we will discuss in more detail in the next subsection. A key property of the collision operator is the conservation of the collision invariants $\psi \in \{1, \boldsymbol{v}, \frac{1}{2}mv^2\}$,

$$\int \psi \, C[f] \; \mathrm{d}^3\boldsymbol{v} = 0. \tag{9}$$

In the BGK model this holds because $f^{(0)}$ is chosen to match the local density, momentum, and energy of f, so that $\int (f - f^{(0)}) \; \mathrm{d}^3\boldsymbol{v} = 0$, $\int \boldsymbol{v}(f - f^{(0)}) \; \mathrm{d}^3\boldsymbol{v} = 0$, and $\int v^2(f - f^{(0)}) \; \mathrm{d}^3\boldsymbol{v} = 0$, i.e., it disappears under the actions of the moments. Let us now derive the evolution equations for some of the low-moment quantities from above.

1.1 Zeroth-moment (mass conservation)

Integrating the Boltzmann equation over $\mathrm{d}^3\boldsymbol{v}$ gives

$$\frac{\partial}{\partial t}\int f\ \mathrm{d}^3\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\int \boldsymbol{v}f\ \mathrm{d}^3\boldsymbol{v} = \underbrace{\int C[f]\ \mathrm{d}^3\boldsymbol{v}}_{=0}, \tag{10}$$

or, using the moment definitions,

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\boldsymbol{u}) = 0, \tag{11}$$

where we can multiply by the species mass to get the standard continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0. \tag{12}$$

1.2 First-moment (momentum conservation)

Multiply the kinetic equation by $m\boldsymbol{v}$ and integrate,

$$\frac{\partial}{\partial t}\left(m\int \boldsymbol{v}f\ \mathrm{d}^3\boldsymbol{v}\right) + \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\left(m\int \boldsymbol{v}\otimes\boldsymbol{v}f\ \mathrm{d}^3\boldsymbol{v}\right) = \underbrace{m\int \boldsymbol{v}\,C[f]\ \mathrm{d}^3\boldsymbol{v}}_{=0}. \tag{13}$$

where $\boldsymbol{v}\otimes\boldsymbol{v} = v_i v_j$. Let us note that

$$v_i v_j = (u_i + c_i)(u_j + c_j) = u_i u_j + u_i c_j + u_j c_i + c_i c_j, \tag{14}$$

and hence the second integral becomes

$$m\int v_i v_j f\ \mathrm{d}^3\boldsymbol{v} = m\int (u_i u_j + u_i c_j + u_j c_i + c_i c_j)\, f\ \mathrm{d}^3\boldsymbol{v}. \tag{15}$$

The first term yields,

$$m\int u_i u_j f\ \mathrm{d}^3\boldsymbol{v} = m u_i u_j\int f\ \mathrm{d}^3\boldsymbol{v} = \rho u_i u_j = \rho\boldsymbol{v}\otimes\boldsymbol{v}, \tag{16}$$

which is the classic quadratic nonlinearity (where turbulence comes from). The second and third term yield,

$$m\int u_i c_j f\ \mathrm{d}^3\boldsymbol{v} = m u_i\int c_j f\ \mathrm{d}^3\boldsymbol{v} = m u_i\int (v_j - u_j) f\ \mathrm{d}^3\boldsymbol{v} = m u_i\left(\int v_j f\ \mathrm{d}^3\boldsymbol{v} - u_j\int f\ \mathrm{d}^3\boldsymbol{v}\right) = 0, \tag{17}$$

which means the first-moment of the velocity fluctuations away from the bulk flow is zero. Hence the only remaining term is

$$m\int c_i c_j f\ \mathrm{d}^3\boldsymbol{v} = P_{ij} \tag{18}$$

which is exactly the definition of the tensorial pressure $\mathbb{P} = P_{ij}$. Substituting this back into Equation (13)

$$\frac{\partial \rho\boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u} + \mathbb{P}) = 0, \tag{19}$$

or equivalently

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\otimes\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}\cdot\mathbb{P}, \tag{20}$$

after using Equation (12).

1.3 Second-moment (energy conservation)

Next we multiply the kinetic equation by $\frac{1}{2}mv^2$ and integrate,

$$\frac{\partial}{\partial t}\int \frac{1}{2}mv^2 f \ \mathrm{d}^3\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \int \frac{1}{2}mv^2 \boldsymbol{v}\, f \ \mathrm{d}^3\boldsymbol{v} = \underbrace{\int \frac{1}{2}mv^2\, C[f] \ \mathrm{d}^3\boldsymbol{v}}_{=0}. \tag{21}$$

Decomposing $v^2 = (\boldsymbol{u}+\boldsymbol{c})^2 = u_iu_i + 2u_ic_i + c_ic_i$ and using $\int c_i f \ \mathrm{d}^3\boldsymbol{v} = 0$ on the u_ic_i terms, as we showed previously, then

$$\frac{\partial}{\partial t}\int \frac{1}{2}mv^2 f \ \mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right), \tag{22}$$

is just the Eulerian derivative of the total energy. We expand the last integral in a similar fashion,

$$\int \frac{1}{2}mv^2\boldsymbol{v}\, f \ \mathrm{d}^3\boldsymbol{v} = \int \frac{1}{2}m\left(u_iu_i + 2u_ic_i + c_ic_i\right)v_j\, f \ \mathrm{d}^3\boldsymbol{v} = \frac{1}{2}\rho u^2\boldsymbol{u}$$
$$+ \rho e + \int \frac{1}{2}mc_ic_iv_jf \ \mathrm{d}^3\boldsymbol{v} + mu_i\int c_iv_jf \ \mathrm{d}^3\boldsymbol{v}. \tag{23}$$

Similar to before, we must expand the tensor product for both integrals,

$$c_iv_j = c_i(u_j + c_j) = c_iu_j + c_ic_j. \tag{24}$$

By substituting this into the first integral,

$$\int \frac{1}{2}mc_ic_iv_jf \ \mathrm{d}^3\boldsymbol{v} = \int \frac{1}{2}mc_i(c_iu_j + c_ic_j)f \ \mathrm{d}^3\boldsymbol{v} = \rho e\boldsymbol{u} + \boldsymbol{q}, \tag{25}$$

by definition, where we remind the reader that the $\boldsymbol{q}$ is the heat-flux. The last integral is

$$mu_i\int c_iv_jf \ \mathrm{d}^3\boldsymbol{v} = mu_i\int (c_iu_j + c_ic_j)f \ \mathrm{d}^3\boldsymbol{v} = mu_i\int c_ic_jf \ \mathrm{d}^3\boldsymbol{v} = u_iP_{ij} = \mathbb{P}\cdot\boldsymbol{u}. \tag{26}$$

Putting this together gives the total energy equation,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \boldsymbol{\nabla}\cdot\left[\left(\frac{1}{2}\rho u^2 + \rho e\right)\boldsymbol{u} + \mathbb{P}\cdot\boldsymbol{u} + \boldsymbol{q}\right] = 0. \tag{27}$$

We have therefore showed that the zeroth, first, and second moment describe the fluid equations that we use regularly to describe the world around us.

2 Chapman-Enskog expansion

The Boltzmann equation contains physics on two very different scales: fast microscopic velocity relaxation driven by collisions, and slow macroscopic evolution of density, velocity, and temperature. The Chapman-Enskog method exploits this separation by expanding about local thermodynamic equilibrium in the small parameter

$$\epsilon \equiv \mathrm{Kn} = \frac{\lambda}{L} \ll 1, \tag{28}$$

i.e., the Knudsen number, Kn, the ratio of the mean free path λ to a characteristic (usually system) macroscopic length scale L. Naturally, for $\mathrm{Kn} \ll 1$ the plasma only has very small deviations away from being in local thermal equilibrium. We therefore seek a solution to the BGK equation in which all space-time dependence of f enters only through the hydrodynamic fields $n(\boldsymbol{x},t), \boldsymbol{u}(\boldsymbol{x},t), T(\boldsymbol{x},t)$, and expand

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots. \tag{29}$$

To maintain consistency between fast collisional relaxation, $\mathcal{O}(1/\epsilon)$, and slow hydrodynamic evolution $\mathcal{O}(\epsilon^{(n)})$, we also introduce a multiscale time derivative

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^{(0)}} + \epsilon\frac{\partial}{\partial t^{(1)}} + \epsilon^2\frac{\partial}{\partial t^{(2)}} + \cdots. \tag{30}$$

Next we equate powers of ϵ in the Boltzmann BGK equation to yield a hierarchy of equations of different $\mathcal{O}(\epsilon^{(n)})$.

2.1 Zeroth-order: local equilibrium and Euler equations

We assume a hydrodynamic ordering in which macroscopic fields vary on timescales long compared to the collision time (noting that $\boldsymbol{v}\cdot\nabla f^{(0)} = O(1)$), so that the leading distribution $f^{(0)}$ evolves only on the slow (hydrodynamic) time. In the multiscale expansion this means

$$\frac{\partial f^{(0)}}{\partial t^{(0)}} = \frac{\partial n}{\partial t^{(0)}} = \frac{\partial \boldsymbol{u}}{\partial t^{(0)}} = \frac{\partial T}{\partial t^{(0)}} = 0. \tag{31}$$

Hence, at leading order $O(\epsilon^0)$ we obtain

$$0 = -\frac{1}{\tau}\left(f^{(0)} - f_M\right), \tag{32}$$

where f_M is the Maxwell-Boltzmann distribution function. We already knew that the local thermal equilibrium would result in

$$f^{(0)} = f_M(n, \boldsymbol{u}, T), \tag{33}$$

i.e. the zeroth-order distribution is the local Maxwellian. Taking the zeroth, first, and second velocity moments of the kinetic equation at this order recovers the Euler equations, as in the previous subsection. However, what we show now is that to $\mathcal{O}(\varepsilon^{(0)})$ the pressure tensor reduces to

$$P_{ij}^{(0)} = p\,\delta_{ij}, \qquad p = nk_BT, \tag{34}$$

and the heat flux vanishes,

$$\boldsymbol{q}^{(0)} = 0, \tag{35}$$

due to the parity and symmetry of $f_M(n, \boldsymbol{u}, T)$.

2.1.1 Isotropic pressure

To zeroth order we define the pressure,

$$P_{ij}^{(0)} \equiv m\int c_i c_j f^{(0)}\ \mathrm{d}^3\boldsymbol{v} = m\int c_i c_j f_M(c)\ \mathrm{d}^3\boldsymbol{c}. \tag{36}$$

where $\mathrm{d}^3\boldsymbol{v} = \mathrm{d}^3\boldsymbol{c}$. Note that $f_M(c)$ is rotationally symmetric, in that it only depends upon $|\boldsymbol{c}|$. Does the rotational symmetry extend itself to the entire integrand? Yes. Let us apply rotation matrices R_{ij} to each of the c_i, such that $c'_i = R_{ij}c_j$ noting that $\mathrm{d}^3\boldsymbol{c} = \det|R_{ij}|\,\mathrm{d}^3\boldsymbol{c}' = \mathrm{d}^3\boldsymbol{c}'$,

$$P_{ij}^{(0)} \equiv m\int c'_i c'_j f^{(0)}(c')\ \mathrm{d}^3\boldsymbol{c}' = mR_{im}R_{jl}\int c_m c_l f^{(0)}(c)\,\mathrm{d}^3\boldsymbol{c} = R_{im}R_{jl}P_{ml}^{(0)}, \tag{37}$$

Since this holds for all rotations R, $P_{ij}^{(0)}$ must satisfy $P = RPR^T$ for all R, which implies $P_{ij}^{(0)} = A\delta_{ij}$. Because

$$P_{kk}^{(0)} = m\int c^2 f_M\ \mathrm{d}^3\boldsymbol{c} = 2\rho e = 3nk_BT, \tag{38}$$

$A = P_{kk}^{(0)}/3 = nk_BT = p$, as expected for scalar pressure.

2.1.2 Vanishing heat flux

It is not hard to imagine that in local thermal equilibrium there is no heat flux. But let us show it explicitly. By definition

$$q_i^{(0)} \equiv \int \frac{1}{2}mc^2\, c_i\, f^{(0)}\ \mathrm{d}^3\boldsymbol{v} = \int \frac{1}{2}mc^2\, c_i\, f_M(c)\ \mathrm{d}^3\boldsymbol{c}. \tag{39}$$

Let us observe the integrand $1/2\,mc^2\,c_i\,f_M(c)$, where c^2 is even in c, c_i is odd in c and $f_M(c)$ is even in c. This makes $1/2\,mc^2\,c_i\,f_M(c)$ and odd function in $\boldsymbol{c}$, and since $\int \mathrm{d}^3\boldsymbol{c}$ is over all of $\mathbb{R}^3$, $q_i = 0$. Substituting these into (20) and (27) yields the inviscid Euler equations for $n, \boldsymbol{u}, T$. These Euler equations determine the zeroth-order time derivatives $\partial_{t_0}n$, $\partial_{t_0}\boldsymbol{u}$, and $\partial_{t_0}T$ that appear in $f^{(1)}$.

2.2 First-order expression for $f^{(1)}$

At order $O(\epsilon^1)$ the BGK equation yields

$$\frac{\partial f^{(0)}}{\partial t_0} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f^{(0)} = -\frac{1}{\tau} f^{(1)}. \tag{40}$$

Thus

$$f^{(1)} = -\tau \left(\frac{\partial f^{(0)}}{\partial t_0} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f^{(0)} \right) = -\tau \left(\frac{\mathrm{d} f^{(0)}}{\mathrm{d} t^{(0)}} + \boldsymbol{c} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f^{(0)} \right). \tag{41}$$

where $\mathrm{d}/\,\mathrm{d}t^{(0)} \equiv \partial_{t_0} + \boldsymbol{u} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}}$ is the (zeroth–order) material derivative and noting that we pick up $\boldsymbol{c} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f_M$ by substituting $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{c}$. Since $f^{(0)} = f_M(n, \boldsymbol{u}, T)$ depends on space and time only through the hydrodynamic fields, the derivatives in (41) are evaluated using the chain rule,

$$\frac{\mathrm{d} f^{(0)}}{\mathrm{d} t^{(0)}} = \frac{\mathrm{d} f_M}{\mathrm{d} t^{(0)}} = \frac{\partial f_M}{\partial n} \frac{\mathrm{d} n}{\mathrm{d} t^{(0)}} + \frac{\partial f_M}{\partial T} \frac{\mathrm{d} T}{\mathrm{d} t^{(0)}} + \frac{\partial f_M}{\partial u_i} \frac{\mathrm{d} u_i}{\mathrm{d} t^{(0)}}, \tag{42}$$

The zeroth-order time derivatives in n, $\boldsymbol{u}$, and T are determined by the Euler equations obtained in the previous section, which I now go through below.

2.2.1 The zeroth-order material derivatives

As an explicit example, consider the zeroth-order density derivative appearing in (42). From the Euler continuity equation (12),

$$\frac{\partial n}{\partial t_0} + \boldsymbol{\nabla} \cdot (n \boldsymbol{u}) = 0, \tag{43}$$

so the material derivative is

$$\frac{\mathrm{d} n}{\mathrm{d} t^{(0)}} \equiv \frac{\partial n}{\partial t_0} + \boldsymbol{u} \cdot \boldsymbol{\nabla} n = -n \, \boldsymbol{\nabla} \cdot \boldsymbol{u}. \tag{44}$$

Thus all occurrences of $\mathrm{d}n/\,\mathrm{d}t^{(0)}$ in (42) can be replaced by spatial gradients of the velocity field. Similarly, the Euler momentum equation

$$\rho \frac{\mathrm{d} \boldsymbol{u}}{\mathrm{d} t^{(0)}} = -\boldsymbol{\nabla} p, \qquad \text{gives} \qquad \frac{\mathrm{d} u_i}{\mathrm{d} t^{(0)}} = -\frac{1}{\rho} \partial_i p, \tag{45}$$

so time derivatives of the bulk velocity are replaced by pressure gradients. Finally, the zeroth-order energy equation implies

$$\frac{\mathrm{d} T}{\mathrm{d} t^{(0)}} = -\frac{2}{3} T \, \boldsymbol{\nabla} \cdot \boldsymbol{u}, \tag{46}$$

for a monatomic ideal gas. Hence, we can substitute these back into (42),

$$\frac{\mathrm{d} f_M}{\mathrm{d} t^{(0)}} = -\frac{\partial f_M}{\partial n} n \, \boldsymbol{\nabla} \cdot \boldsymbol{u} - \frac{\partial f_M}{\partial T} \frac{2}{3} T \, \boldsymbol{\nabla} \cdot \boldsymbol{u} - \frac{\partial f_M}{\partial u_i} \frac{1}{\rho} \partial_i p, \tag{47}$$

2.2.2 Partial Derivatives of the Maxwellian

We write the local Maxwellian as

$$f_M(n, \boldsymbol{u}, T) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\!\left(-\frac{m c^2}{2 k_B T} \right), \qquad \boldsymbol{c} \equiv \boldsymbol{v} - \boldsymbol{u}. \tag{48}$$

It is convenient to differentiate $\ln f_M$ and then multiply by f_M. Taking logarithms,

$$\ln f_M = \ln n + \frac{3}{2} \ln\!\left(\frac{m}{2\pi k_B T} \right) - \frac{m c^2}{2 k_B T}. \tag{49}$$

Derivative with respect to density. Since f_M is linear in n,

$$\frac{\partial f_M}{\partial n} = \frac{f_M}{n}. \tag{50}$$

Derivative with respect to temperature. Differentiating $\ln f_M$ with respect to T (holding n and $\boldsymbol{u}$ fixed) gives

$$\frac{\partial \ln f_M}{\partial T} = -\frac{3}{2}\frac{1}{T} + \frac{mc^2}{2k_B T^2}, \qquad \text{hence} \qquad \frac{\partial f_M}{\partial T} = f_M\left(-\frac{3}{2}\frac{1}{T} + \frac{mc^2}{2k_B T^2}\right). \tag{51}$$

Derivative with respect to the bulk velocity. Using $c^2 = (\boldsymbol{v} - \boldsymbol{u})^2$,

$$\frac{\partial c^2}{\partial u_i} = \frac{\partial}{\partial u_i}\left[(v_j - u_j)(v_j - u_j)\right] = -2c_i, \qquad \text{so} \qquad \frac{\partial \ln f_M}{\partial u_i} = -\frac{\partial}{\partial u_i}\left(\frac{mc^2}{2k_B T}\right) = \frac{m}{k_B T}c_i, \tag{52}$$

and therefore

$$\frac{\partial f_M}{\partial u_i} = \frac{m}{k_B T} c_i f_M. \tag{53}$$

Substituting these expressions into the chain rule (47) yields

$$\frac{\mathrm{d} f_M}{\mathrm{d} t^{(0)}} = -\frac{m}{k_B T}\left(\frac{c^2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{c_i}{\rho}\partial_i p\right) f_M, \tag{54}$$

Gradient of the streaming term. Finally, the streaming contribution may be written in the same spirit,

$$\boldsymbol{c}\cdot\boldsymbol{\nabla} f_M = f_M\, \boldsymbol{c}\cdot\boldsymbol{\nabla}\ln f_M. \tag{55}$$

To proceed we compute $\boldsymbol{\nabla}\ln f_M$ explicitly. From

$$\ln f_M = \ln n + \frac{3}{2}\ln\left(\frac{m}{2\pi k_B T}\right) - \frac{mc^2}{2k_B T}, \tag{56}$$

we obtain

$$\begin{aligned} \boldsymbol{\nabla}\ln f_M &= \frac{\boldsymbol{\nabla} n}{n} - \frac{3}{2}\frac{\boldsymbol{\nabla} T}{T} - \boldsymbol{\nabla}\left(\frac{mc^2}{2k_B T}\right) = \frac{\boldsymbol{\nabla} n}{n} - \frac{3}{2}\frac{\boldsymbol{\nabla} T}{T} - \frac{m}{2k_B}\boldsymbol{\nabla}\left(\frac{c^2}{T}\right) \\ &= \frac{\boldsymbol{\nabla} n}{n} - \frac{3}{2}\frac{\boldsymbol{\nabla} T}{T} - \frac{m}{2k_B}\left(\frac{\boldsymbol{\nabla} c^2}{T} - \frac{c^2}{T^2}\boldsymbol{\nabla} T\right) = \frac{\boldsymbol{\nabla} n}{n} + \left(\frac{mc^2}{2k_B T} - \frac{3}{2}\right)\frac{\boldsymbol{\nabla} T}{T} - \frac{m}{2k_B T}\boldsymbol{\nabla} c^2. \end{aligned} \tag{57}$$

It remains to evaluate $\boldsymbol{\nabla} c^2$. Since $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{v}$ is an independent phase-space variable,

$$\partial_j c_i = -\partial_j u_i, \tag{58}$$

and therefore

$$\partial_j c^2 = \partial_j(c_i c_i) = 2c_i \partial_j c_i = -2c_i \partial_j u_i, \tag{59}$$

Contracting (57) with $\boldsymbol{c}$ and using (59) gives

$$\boldsymbol{c}\cdot\boldsymbol{\nabla}\ln f_M = c_j\partial_j \ln n + \left(\frac{mc^2}{2k_B T} - \frac{3}{2}\right) c_j \partial_j \ln T + \frac{m}{k_B T} c_i c_j\, \partial_j u_i. \tag{60}$$

Hence the streaming contribution can be written as

$$\boldsymbol{c}\cdot\boldsymbol{\nabla} f_M = f_M\left[c_j\partial_j \ln n + \left(\frac{mc^2}{2k_B T} - \frac{3}{2}\right) c_j \partial_j \ln T + \frac{m}{k_B T} c_i c_j\, \partial_j u_i\right]. \tag{61}$$

Putting it all back together. By substituting (54) and (61) into (41), the first-order distribution function is then,

$$f^{(1)} = -\tau\left(\frac{\mathrm{d} f_M}{\mathrm{d} t^{(0)}} + \boldsymbol{c}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} f^{(0)}\right), \tag{62}$$

$$\frac{\mathrm{d} f_M}{\mathrm{d} t^{(0)}} = -\frac{m}{k_B T}\left(\frac{c^2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{c_i}{\rho}\partial_i p\right) f_M, \tag{63}$$

$$\boldsymbol{c}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} f_M = f_M\left[c_j\partial_j \ln n + \left(\frac{mc^2}{2k_B T} - \frac{3}{2}\right) c_j \partial_j \ln T + \frac{m}{k_B T} c_i c_j\, \partial_j u_i\right], \tag{64}$$

finally providing a close-form solution to $f^{(1)}$, based only on $f^{(0)} = f_M$.

3 Moments of $f^{(1)}$: heat flux and pressure corrections

We now connect the Chapman-Enskog expansion

$$f = f^{(0)} + \epsilon f^{(1)} + O(\epsilon^2), \tag{65}$$

directly to macroscopic transport by taking velocity moments of the distribution function. At zeroth order $f^{(0)} = f_M$ gives isotropic pressure and vanishing heat flux,

$$P_{ij}^{(0)} = m \int c_i c_j f_M \, \mathrm{d}^3\boldsymbol{c} = p\,\delta_{ij}, \qquad q_i^{(0)} = \int \frac{1}{2} m c^2 c_i f_M \, \mathrm{d}^3\boldsymbol{c} = 0, \tag{66}$$

as we showed earlier. All irreversible transport therefore enters through the first-order correction $f^{(1)}$.

3.1 Relevant first-order fluxes

We define the first-order pressure tensor and heat flux as

$$P_{ij}^{(1)} \equiv m \int c_i c_j f^{(1)} \, \mathrm{d}^3\boldsymbol{c}, \qquad q_i^{(1)} \equiv \int \frac{1}{2} m c^2 c_i f^{(1)} \, \mathrm{d}^3\boldsymbol{c}. \tag{67}$$

The full stress tensor is then

$$P_{ij} = P_{ij}^{(0)} + \epsilon P_{ij}^{(1)} + O(\epsilon^2). \tag{68}$$

Further, we define the viscous stress as the deviation from the isotropic pressure,

$$\pi_{ij} \equiv P_{ij} - p\,\delta_{ij} = \epsilon P_{ij}^{(1)} + O(\epsilon^2). \tag{69}$$

3.2 Decomposition of $f^{(1)}$ by tensorial parity

The first-order correction obtained from (62) may be grouped by its velocity dependence into

$$f^{(1)} = f_{\text{odd}}^{(1)} + f_{\text{bulk}}^{(1)} + f_{\text{shear}}^{(1)}, \tag{70}$$

where,

- $f_{\text{odd}}^{(1)}$ is odd in $\boldsymbol{c}$ (typically $\propto c_j \partial_j \ln n$ and $c_j \partial_j \ln T$),
- $f_{\text{bulk}}^{(1)}$ is isotropic in indices but even in $\boldsymbol{c}$ (typically $\propto c^2 \boldsymbol{\nabla} \cdot \boldsymbol{u}$),
- $f_{\text{shear}}^{(1)}$ is even and traceless-quadratic in $\boldsymbol{c}$ (typically $\propto c_i c_j$ contracted with velocity gradients).

This decomposition is useful because different moments "select" terms with different symmetries from (62). What this amounts to is organizing different macroscopic forces by how they act to restore each fluid element to local thermal equilibrium.

3.3 First-order heat flux from odd terms of $f^{(1)}$ and Fourier's law of heat conduction

From the full first–order correction (62), the terms that are odd in $\boldsymbol{c}$ are those proportional to $c_j \partial_j \ln n$, $c_j \partial_j \ln T$, and $c_i \partial_i p$. Collecting them, the odd part of $f^{(1)}$ can be written as

$$f_{\text{odd}}^{(1)} = -\tau f_M \left[c_j \partial_j \ln n + \left(\frac{mc^2}{2k_B T} - \frac{3}{2} \right) c_j \partial_j \ln T - \frac{m}{k_B T} \frac{c_i}{\rho} \partial_i p \right]. \tag{71}$$

Substituting this into the definition of the heat flux gives

$$q_i^{(1)} = -\tau \int \frac{1}{2} m c^2 c_i \left[c_j \partial_j \ln n + \left(\frac{mc^2}{2k_B T} - \frac{3}{2} \right) c_j \partial_j \ln T - \frac{m}{k_B T} \frac{c_j}{\rho} \partial_j p \right] f_M \, \mathrm{d}^3\boldsymbol{c}. \tag{72}$$

By isotropy of the Maxwellian, integrals of the form $\int c_i c_j F(c^2) f_M \; \mathrm{d}^3\boldsymbol{c}$ reduce to

$$\int c_i c_j F(c^2) f_M \; \mathrm{d}^3\boldsymbol{c} = \frac{1}{3}\delta_{ij} \int c^2 F(c^2) f_M \; \mathrm{d}^3\boldsymbol{c}. \tag{73}$$

Therefore,

$$\begin{aligned} q_i^{(1)} &= -\tau\, \partial_i \ln n \, \frac{1}{3} \int \frac{1}{2} m c^4 f_M \; \mathrm{d}^3\boldsymbol{c} - \tau\, \partial_i \ln T \, \frac{1}{3} \int \frac{1}{2} m \left(\frac{mc^2}{2k_B T} - \frac{3}{2} \right) c^4 f_M \; \mathrm{d}^3\boldsymbol{c} \\ &\quad + \tau\, \partial_i p \, \frac{1}{3} \frac{m}{k_B T} \frac{1}{\rho} \int \frac{1}{2} m c^4 f_M \; \mathrm{d}^3\boldsymbol{c}. \end{aligned} \tag{74}$$

The fourth moment of f_M in $\boldsymbol{c}$ is (which can be shown by direct integration),

$$\int c^4 f_M \; \mathrm{d}^3\boldsymbol{c} = 15n \left(\frac{k_B T}{m} \right)^2, \tag{75}$$

so

$$\int \frac{1}{2} m c^4 f_M \; \mathrm{d}^3\boldsymbol{c} = \frac{15}{2} n \frac{(k_B T)^2}{m}. \tag{76}$$

Using $p = nk_B T$ and $\rho = mn$, the density-gradient and pressure-gradient terms are then

$$-\tau\, \partial_i \ln n \, \frac{1}{3} \int \frac{1}{2} m c^4 f_M \; \mathrm{d}^3\boldsymbol{c} = -\tau\, \partial_i \ln n \, \frac{1}{3} \frac{15}{2} n \frac{(k_B T)^2}{m} = -\tau \partial_i n \frac{5}{2} \frac{(k_B T)^2}{m} \tag{77}$$

and

$$\tau\, \partial_i p \, \frac{1}{3} \frac{m}{k_B T} \frac{1}{\rho} \int \frac{1}{2} m c^4 f_M \; \mathrm{d}^3\boldsymbol{c} = \tau\, \partial_i p \, \frac{1}{k_B T} \frac{5}{2} \frac{(k_B T)^2}{m} = \tau\, \partial_i n \, \frac{5}{2} \frac{(k_B T)^2}{m} + \tau\, n \partial_i \ln T \, \frac{5}{2} \frac{(k_B T)^2}{m} \tag{78}$$

and hence the first term of the pressure-gradient cancels exactly with the density gradient. This leaves only the temperature gradient terms in $q_i^{(1)}$,

$$q_i^{(1)} = \tau\, n \partial_i \ln T \, \frac{5}{2} \frac{(k_B T)^2}{m} - \tau\, \partial_i \ln T \, \frac{1}{3} \int \frac{1}{2} m \left(\frac{mc^2}{2k_B T} - \frac{3}{2} \right) c^4 f_M \; \mathrm{d}^3\boldsymbol{c}. \tag{79}$$

Let us now break the final integral into two. Starting with the simplest term is the $-3/2$ term in the final integral, resembling again the fourth moment of f_M, which we already calculated above. Hence,

$$-\tau\, \partial_i \ln T \, \frac{1}{3} \int \frac{1}{2} m \left(-\frac{3}{2} \right) c^4 f_M \; \mathrm{d}^3\boldsymbol{c}. = \tau \partial_i \ln T \frac{1}{2} \int \frac{1}{2} m c^4 f_M d^3 c = \tau n \partial_i \ln T \frac{15}{4} \frac{(k_B T)^2}{m}, \tag{80}$$

hence,

$$q_i^{(1)} = \tau\, n \partial_i \ln T \, \frac{25}{4} \frac{(k_B T)^2}{m} - \tau\, \partial_i \ln T \, \frac{m^2}{12 k_B T} \int c^6 f_M \; \mathrm{d}^3\boldsymbol{c}, \tag{81}$$

leaving us with a final sixth moment of f_M to calculate. One can show that,

$$\int c^6 f_M \; \mathrm{d}^3\boldsymbol{c} = 105\, n \left(\frac{k_B T}{m} \right)^3, \tag{82}$$

and so

$$-\tau\, \partial_i \ln T \, \frac{m^2}{12 k_B T} \int c^6 f_M \; \mathrm{d}^3\boldsymbol{c} = -\tau\, \partial_i \ln T \, \frac{m^2}{12 k_B T} 105\, n \left(\frac{k_B T}{m} \right)^3 = -\tau n\, \partial_i \ln T \, \frac{105}{12} \frac{(k_B T)^2}{m}. \tag{83}$$

This gives,

$$q_i^{(1)} = -\tau\, n\, \partial_i \ln T \, \frac{5}{2} \frac{(k_B T)^2}{m} = -\tau \, \frac{5}{2} \frac{n k_B^2 T}{m} \partial_i T = -\tau p \, \frac{5}{2} \frac{k_B}{m} \partial_i T, \tag{84}$$

i.e.,

$$\boldsymbol{q}^{(1)} = -\kappa \boldsymbol{\nabla} T, \qquad \kappa = \frac{5}{2} \frac{k_B}{m} p \tau. \tag{85}$$

Thus the odd part of the first order Chapman-Enskog correction produces Fourier's law of heat conduction, with thermal conductivity proportional to the collisional relaxation time τ.

3.4 Pressure corrections and bulk viscosity from the isotropic, even terms of $f^{(1)}$

The first-order pressure tensor is

$$P_{ij}^{(1)} = m \int c_i c_j f^{(1)} \, \mathrm{d}^3\boldsymbol{c}. \tag{86}$$

Since $c_i c_j$ is even in $\boldsymbol{c}$, any odd part of $f^{(1)}$ (the terms responsible for $\boldsymbol{q}^{(1)}$) integrates to zero by parity. Thus only the even-in-$\boldsymbol{c}$ pieces of $f^{(1)}$ contribute to $P_{ij}^{(1)}$. From the full first-order correction (62), the even contributions that are *isotropic* in velocity space are those proportional to c^2 (rather than $c_i c_j$), and they multiply the scalar compression $\boldsymbol{\nabla} \cdot \boldsymbol{u}$. Denote this part by

$$f_{\mathrm{bulk}}^{(1)} = -\tau f_M \frac{m}{k_B T} \left(\frac{c^2}{3} \boldsymbol{\nabla} \cdot \boldsymbol{u} \right), \tag{87}$$

where the factor $1/3$ is convenient because $c^2/3$ is the isotropic part of the quadratic tensor $c_i c_j$. Inserting (87) into the definition of $P_{ij}^{(1)}$:

$$P_{ij,\,\mathrm{bulk}}^{(1)} = m \int c_i c_j f_{\mathrm{bulk}}^{(1)} \, \mathrm{d}^3\boldsymbol{c} = -\tau \, \boldsymbol{\nabla} \cdot \boldsymbol{u} \, \frac{m^2}{3 k_B T} \int c_i c_j c^2 f_M \, \mathrm{d}^3\boldsymbol{c}. \tag{88}$$

By isotropy of the Maxwellian, the tensor integral must be proportional to δ_{ij}:

$$\int c_i c_j c^2 f_M \, \mathrm{d}^3\boldsymbol{c} = \frac{1}{3} \delta_{ij} \int c^4 f_M \, \mathrm{d}^3\boldsymbol{c}. \tag{89}$$

Using the fourth moment already computed in the heat-flux section,

$$\int c^4 f_M \, \mathrm{d}^3\boldsymbol{c} = 15 n \left(\frac{k_B T}{m} \right)^2, \tag{90}$$

we obtain

$$\int c_i c_j c^2 f_M \, \mathrm{d}^3\boldsymbol{c} = \frac{1}{3} \delta_{ij} \, 15 n \left(\frac{k_B T}{m} \right)^2 = 5 n \left(\frac{k_B T}{m} \right)^2 \delta_{ij}. \tag{91}$$

Substituting back into (88) gives

$$P_{ij,\,\mathrm{bulk}}^{(1)} = -\tau \, \boldsymbol{\nabla} \cdot \boldsymbol{u} \, \frac{m^2}{3 k_B T} \, 5 n \left(\frac{k_B T}{m} \right)^2 \delta_{ij} = -\tau \, \boldsymbol{\nabla} \cdot \boldsymbol{u} \, \frac{5}{3} \, n k_B T \, \delta_{ij}. \tag{92}$$

Since $p = n k_B T$, this can be written compactly as

$$P_{ij,\,\mathrm{bulk}}^{(1)} = -\zeta \, (\boldsymbol{\nabla} \cdot \boldsymbol{u}) \, \delta_{ij}, \qquad \zeta \equiv \frac{5}{3} p \tau. \tag{93}$$

Interpretation (bulk viscosity and equation of state). Equation (93) is an *isotropic* (pure-trace) correction to the pressure tensor. It therefore does not contribute to shear stresses; rather, it modifies the relation between isotropic stress and the local compression $\boldsymbol{\nabla} \cdot \boldsymbol{u}$ and is naturally interpreted as a bulk-viscosity-type term. A caution is warranted. For a monatomic ideal gas with only elastic binary collisions, the physical bulk viscosity vanishes in the full Boltzmann theory. In the Chapman-Enskog expansion this occurs because any isotropic $O(\epsilon)$ correction to the pressure tensor corresponds to a change in the local energy density and can therefore be absorbed into a redefinition of the temperature. As a result, only the traceless (deviatoric) part of the stress tensor represents a true first-order dissipative effect.

3.5 Shear viscosity from the traceless, even terms of $f^{(1)}$.

We now compute the deviatoric (shear) stress arising from the traceless, even terms in $f^{(1)}$. Since the pressure tensor is even in $\boldsymbol{c}$,

$$P_{ij}^{(1)} = m \int c_i c_j f^{(1)} \, \mathrm{d}^3\boldsymbol{c}, \tag{94}$$

the odd part of $f^{(1)}$ does not contribute. Moreover, we have already isolated the *isotropic*, components are proportional to $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ (the bulk/trace correction). We therefore focus only on the traceless part of the velocity gradient, introducing the standard decomposition

$$\partial_j u_i = S_{ij} + \Omega_{ij} + \frac{1}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{95}$$

with

$$S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) - \frac{1}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}, \qquad \Omega_{ij} = \frac{1}{2}(\partial_j u_i - \partial_i u_j), \tag{96}$$

so that $S_{ii} = 0$. In the Chapman-Enskog correction, the part that produces shear stress must be (i) even in $\boldsymbol{c}$ and (ii) traceless in the indices contracted with P_{ij}. This therefore corresponds to only the $c_i c_j \partial_j u_i$ term in (54), which we can write in traceless form $\left(c_i c_j - \frac{1}{3}c^2\delta_{ij}\right)$ and the rate-of-strain S_{ij}, giving

$$f_{\text{shear}}^{(1)} = -\tau f_M \frac{m}{k_B T}\left(c_i c_j - \frac{1}{3}c^2\delta_{ij}\right) S_{ij}. \tag{97}$$

(The antisymmetric part Ω_{ij} cannot contribute because it contracts to zero with the symmetric tensor $c_i c_j$.) The corresponding first-order pressure correction is

$$P_{ij,\,\text{shear}}^{(1)} = m \int c_i c_j f_{\text{shear}}^{(1)} \, \mathrm{d}^3\boldsymbol{c} = -\tau \frac{m^2}{k_B T} S_{k\ell} \int c_i c_j \left(c_k c_\ell - \frac{1}{3}c^2\delta_{k\ell}\right) f_M \, \mathrm{d}^3\boldsymbol{c}. \tag{98}$$

By isotropy of the Maxwellian, the fourth moment in $\boldsymbol{c}$ has the general form

$$\int c_i c_j c_k c_\ell f_M \, \mathrm{d}^3\boldsymbol{c} = A\left(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right), \tag{99}$$

where A is fixed by contracting indices:

$$\int c^4 f_M \, \mathrm{d}^3\boldsymbol{c} = \int c_i c_i c_j c_j f_M \, \mathrm{d}^3\boldsymbol{c} = 15A \qquad \Rightarrow \qquad A = \frac{1}{15}\int c^4 f_M \, \mathrm{d}^3\boldsymbol{c}. \tag{100}$$

Using the previously evaluated f_M moment in $\boldsymbol{c}$

$$\int c^4 f_M \, \mathrm{d}^3\boldsymbol{c} = 15n\left(\frac{k_B T}{m}\right)^2, \tag{101}$$

we obtain $A = n\left(\frac{k_B T}{m}\right)^2$. Substituting into (98) and using $S_{kk} = 0$ yields

$$\int c_i c_j \left(c_k c_\ell - \frac{1}{3}c^2\delta_{k\ell}\right) f_M \, \mathrm{d}^3\boldsymbol{c} = A\left(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right), \tag{102}$$

$$\Rightarrow \quad P_{ij,\,\text{shear}}^{(1)} = -\tau\frac{m^2}{k_B T} A\left(S_{ij} + S_{ji}\right) = -2\tau\frac{m^2}{k_B T} A\, S_{ij}. \tag{103}$$

Finally, since $A = n\left(\frac{k_B T}{m}\right)^2$ and $p = nk_B T$, we find

$$P_{ij,\,\text{shear}}^{(1)} = -2p\tau\, S_{ij}. \tag{104}$$

Thus the deviatoric stress tensor, π_{ij}, is

$$\pi_{ij} \equiv P_{ij} - p\delta_{ij} = -2\mu\, S_{ij}, \qquad \mu = p\tau, \tag{105}$$

which is the Newtonian shear stress with dynamic viscosity coefficient $\mu = p\tau$, derived directly from the Chapman-Enskog expansion of the Boltzmann equation with the BGK collision operator.