Fluid moments and linear waves in ideal magnetohydrodynamics

James R. Beattie

February 6, 2026

Contents

1 Lecture goals

This lecture is split into two parts. The first, which pertains to understanding the relationship between a microscopic statistical description of individual particles and a macroscopic fluid model for such a system, and the second which describes how to calculate and get information about a magnetized fluid model under very small perturbations away from a state of homogeneity ($\boldsymbol{\nabla}_{\boldsymbol{x}} = 0$) and stationarity ($\partial_t = 0$). The goals of this lecture are as follows:

1. *To develop an intuition for the connection between the phase space,* $\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{v}$ *distribution function,* f*, for a species of particles and the corresponding fluid theory.* The principle goal is, given an f, you can derive a fluid theory (under the standard assumptions).

2. *To understand how to linearize and interpret linear modes in a fluid model*, with focus on how to solve a for a set of eigenmodes of the linear system, and to understand concepts like wave propagation, phase velocity, and compressibility of the modes.

The philosophy of this lecture is to provide simple cases, for which the anyone may extend to the cases (e.g., relativistic) that torment them and their research work.

2 Boltzmann equation for a monoatomic gas

In its nonrelativistic form, the Boltzmann equation, describing the phase-space, $(\boldsymbol{x}, \boldsymbol{v})$, evolution of the distribution function, f, including external forces and collisions, reads

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f + \frac{\boldsymbol{F}(\boldsymbol{x}, t)}{m} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} f = C[f], \tag{1}$$

where $\boldsymbol{\nabla}_{\boldsymbol{x}}$ and $\boldsymbol{\nabla}_{\boldsymbol{v}}$ denote gradients in coordinate space and velocity space, respectively, and $\boldsymbol{F}$ is the external force (e.g. gravity or the Lorentz force in a plasma). In the first section of these notes we assume a neutral, single-species gas, each particle with mass m, in an inertial frame with no external forces, $\boldsymbol{F} = \boldsymbol{0}$, so the streaming operator reduces to $\partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f$. We replace the full collision operator by the Bhatnagar-Gross-Krook (BGK) model,

$$C[f] = -\frac{1}{\tau}\left(f - f^{(0)}\right), \tag{2}$$

which assumes that collisions drive the distribution toward a local Maxwellian $f^{(0)}$ on a characteristic relaxation time τ. The BGK operator preserves mass, momentum, and energy while providing the correct tensorial structure of viscous transport. The Boltzmann equation therefore becomes

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} f = -\frac{1}{\tau}\left(f - f^{(0)}\right). \tag{3}$$

The local thermal equilibrium (LTE) distribution $f^{(0)}$ is the Maxwell-Boltzmann distribution

$$f^{(0)}(\boldsymbol{x}, \boldsymbol{v}, t) = n(\boldsymbol{x}, t)\left(\frac{m}{2\pi k_B T(\boldsymbol{x}, t)}\right)^{3/2} \exp\left[-\frac{m|\boldsymbol{v} - \boldsymbol{u}(\boldsymbol{x}, t)|^2}{2k_B T(\boldsymbol{x}, t)}\right], \tag{4}$$

where

$$n(\boldsymbol{x}, t), \qquad \boldsymbol{u}(\boldsymbol{x}, t), \qquad T(\boldsymbol{x}, t), \tag{5}$$

are the number density, bulk flow velocity, and temperature. In these lectures we will assume that f is exactly $f^{(0)}$, i.e., that f has collisionally relaxed to LTE. Note that $f(\boldsymbol{x}, \boldsymbol{v}, t)\,\mathrm{d}^3\boldsymbol{v} = f(\boldsymbol{x}, v, t) 4\pi v^2\,\mathrm{d}v$ is the standard Maxwell-Boltzmann in magnitude of v. These macroscopic fluid fields are defined as velocity moments of f,

$$n \equiv \int f\ \mathrm{d}^3\boldsymbol{v}, \qquad n\boldsymbol{u} \equiv \int \boldsymbol{v} f\ \mathrm{d}^3\boldsymbol{v}, \qquad \frac{3}{2} n k_B T \equiv \int \frac{1}{2} m|\boldsymbol{v} - \boldsymbol{u}|^2 f\ \mathrm{d}^3\boldsymbol{v}. \tag{6}$$

It is convenient to introduce the mass density $\rho \equiv mn$, the peculiar velocity $\boldsymbol{c} \equiv \boldsymbol{v} - \boldsymbol{u}$, and kinetic and internal energy,

$$\frac{1}{2}\rho u^2 \equiv \int \frac{1}{2} m v^2 f \, \mathrm{d}^3\boldsymbol{v}, \qquad \rho e \equiv \int \frac{1}{2} m c^2 f \, \mathrm{d}^3\boldsymbol{v} = \frac{3}{2} n k_B T, \tag{7}$$

for a monatomic gas. We also define the kinetic energy, pressure (stress) tensor and heat flux by

$$P_{ij} \equiv m \int c_i c_j f \, \mathrm{d}^3\boldsymbol{v}, \qquad q_i \equiv \int \frac{1}{2} m c^2 c_i \, f \, \mathrm{d}^3\boldsymbol{v}. \tag{8}$$

For a local Maxwellian, $P_{ij} = p\,\delta_{ij}$ and $q_i = 0$ with $p = n k_B T$, which we will discuss in more detail in the next subsection. A key property of the collision operator is the conservation of the collision invariants $\psi \in \{1, \boldsymbol{v}, \frac{1}{2} m v^2\}$,

$$\int \psi \, C[f] \, \mathrm{d}^3\boldsymbol{v} = 0. \tag{9}$$

In the BGK model this holds because $f^{(0)}$ is chosen to match the local density, momentum, and energy of f, so that $\int (f - f^{(0)}) \, \mathrm{d}^3\boldsymbol{v} = 0$, $\int \boldsymbol{v}(f - f^{(0)}) \, \mathrm{d}^3\boldsymbol{v} = 0$, and $\int v^2 (f - f^{(0)}) \, \mathrm{d}^3\boldsymbol{v} = 0$, i.e., it disappears under the actions of the moments. Let us now derive the evolution equations for some of the low-moment quantities from above.

2.1 Zeroth-moment (mass conservation)

Integrating the Boltzmann equation over $\mathrm{d}^3\boldsymbol{v}$ gives

$$\frac{\partial}{\partial t} \int f \, \mathrm{d}^3\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \int \boldsymbol{v} f \, \mathrm{d}^3\boldsymbol{v} = \underbrace{\int C[f] \, \mathrm{d}^3\boldsymbol{v}}_{=0}, \tag{10}$$

or, using the macroscopic definitions,

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\boldsymbol{u}) = 0, \tag{11}$$

where we can multiply by the species mass to get the standard continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0. \tag{12}$$

Note that $\int v_j \, \partial_{x_j} f \, \mathrm{d}^3\boldsymbol{v} = \partial_{x_j} \int v_j f \, \mathrm{d}^3\boldsymbol{v}$, because $\boldsymbol{v}$ and $\boldsymbol{x}$ are independent phase-space coordinates.

2.2 First-moment (momentum conservation)

Multiply the kinetic equation by $m\boldsymbol{v}$ and integrate,

$$\frac{\partial}{\partial t}\left(m \int \boldsymbol{v} f \, \mathrm{d}^3\boldsymbol{v} \right) + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \left(m \int \boldsymbol{v} \otimes \boldsymbol{v} f \, \mathrm{d}^3\boldsymbol{v} \right) = \underbrace{m \int \boldsymbol{v} \, C[f] \, \mathrm{d}^3\boldsymbol{v}}_{=0}. \tag{13}$$

where $\boldsymbol{v} \otimes \boldsymbol{v} = v_i v_j$. Let us note that

$$v_i v_j = (u_i + c_i)(u_j + c_j) = u_i u_j + u_i c_j + u_j c_i + c_i c_j, \tag{14}$$

and hence the second integral becomes

$$m \int v_i v_j f \, \mathrm{d}^3\boldsymbol{v} = m \int (u_i u_j + u_i c_j + u_j c_i + c_i c_j) \, f \, \mathrm{d}^3\boldsymbol{v}. \tag{15}$$

The first term yields,

$$m \int u_i u_j f \, \mathrm{d}^3\boldsymbol{v} = m u_i u_j \int f \, \mathrm{d}^3\boldsymbol{v} = \rho u_i u_j = \rho \boldsymbol{v} \otimes \boldsymbol{v}, \tag{16}$$

which is the classic quadratic nonlinearity (where turbulence comes from). The second and third term yield,

$$m \int u_i c_j f \ \mathrm{d}^3\boldsymbol{v} = m u_i \int c_j f \ \mathrm{d}^3\boldsymbol{v} = m u_i \int (v_j - u_j) f \ \mathrm{d}^3\boldsymbol{v} = m u_i \left(\int v_j f \ \mathrm{d}^3\boldsymbol{v} - u_j \int f \ \mathrm{d}^3\boldsymbol{v} \right) = 0, \tag{17}$$

which means the first-moment of the velocity fluctuations away from the bulk flow is zero. Hence the only remaining term is

$$m \int c_i c_j f \ \mathrm{d}^3\boldsymbol{v} = P_{ij} \tag{18}$$

which is exactly the definition of the tensorial pressure $\mathbb{P} = P_{ij}$. Substituting this back into Equation (13)

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + \mathbb{P}) = 0, \tag{19}$$

or equivalently

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \otimes \boldsymbol{u} = -\frac{1}{\rho} \boldsymbol{\nabla} \cdot \mathbb{P}, \tag{20}$$

after using Equation (12).

2.3 Second-moment (energy conservation)

Next we multiply the kinetic equation by $\frac{1}{2} m v^2$ and integrate,

$$\frac{\partial}{\partial t} \int \frac{1}{2} m v^2 f \ \mathrm{d}^3\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \int \frac{1}{2} m v^2 \boldsymbol{v} \, f \ \mathrm{d}^3\boldsymbol{v} = \underbrace{\int \frac{1}{2} m v^2 \, C[f] \ \mathrm{d}^3\boldsymbol{v}}_{=0} . \tag{21}$$

Decomposing $v^2 = (\boldsymbol{u} + \boldsymbol{c})^2 = u_i u_i + 2 u_i c_i + c_i c_i$ and using $\int c_i f \, d^3 v = 0$ on the $u_i c_i$ terms, as we showed previously, then

$$\frac{\partial}{\partial t} \int \frac{1}{2} m v^2 f \ \mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 + \rho e \right), \tag{22}$$

is just the Eulerian derivative of the total energy. We expand the last integral in a similar fashion,

$$\begin{aligned} \int \frac{1}{2} m v^2 \boldsymbol{v} \, f \, d^3 v = \int \frac{1}{2} m \left(u_i u_i + 2 u_i c_i + c_i c_i \right) v_j \, f \ \mathrm{d}^3\boldsymbol{v} = \frac{1}{2} \rho u^2 \boldsymbol{u} \\ + \rho e + \int \frac{1}{2} m c_i c_i v_j f \ \mathrm{d}^3\boldsymbol{v} + m u_i \int c_i v_j f \ \mathrm{d}^3\boldsymbol{v}. \end{aligned} \tag{23}$$

Similar to before, we must expand the tensor product for both integrals,

$$c_i v_j = c_i (u_j + c_j) = c_i u_j + c_i c_j . \tag{24}$$

By substituting this into the first integral,

$$\int \frac{1}{2} m c_i c_i v_j f \ \mathrm{d}^3\boldsymbol{v} = \int \frac{1}{2} m c_i (c_i u_j + c_i c_j) f \ \mathrm{d}^3\boldsymbol{v} = \rho e \boldsymbol{u} + \boldsymbol{q}, \tag{25}$$

by definition, where we remind the reader that the $\boldsymbol{q}$ is the heat-flux. The last integral is

$$m u_i \int c_i v_j f \ \mathrm{d}^3\boldsymbol{v} = m u_i \int (c_i u_j + c_i c_j) f \ \mathrm{d}^3\boldsymbol{v} = m u_i \int c_i c_j f \ \mathrm{d}^3\boldsymbol{v} = u_i P_{ij} = \mathbb{P} \cdot \boldsymbol{u}. \tag{26}$$

Putting this together gives the total energy equation,

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 + \rho e \right) + \boldsymbol{\nabla} \cdot \left[\left(\frac{1}{2} \rho u^2 + \rho e \right) \boldsymbol{u} + \mathbb{P} \cdot \boldsymbol{u} + \boldsymbol{q} \right] = 0. \tag{27}$$

We have therefore showed that the zeroth, first, and second moment describe the fluid equations that we use regularly to describe the world around us.

2.4 Isotropic pressure and moment closure

At some point one must ask: how many moments of the distribution function are required to obtain a useful fluid description? In principle, one may continue taking velocity moments of the Boltzmann equation indefinitely, thereby generating an infinite hierarchy of coupled evolution equations. A practical fluid theory therefore requires a closure, truncating the infinite hierarchy.

For a plasma in which collisions rapidly relax the distribution function to a local f_M, the distribution is completely specified by the lowest moments: the n, $\boldsymbol{u}$, and T. In this case, higher-order moments such as the heat-flux tensor, $\mathbb{Q}$, are all zero (it is not hard to imagine that the heat-flux is zero in LTE). An important consequence of frequent collisions rapidly relaxing f to f_M is that f becomes isotropic in $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{u}$. This means,

$$P_{ij} \equiv m \int c_i c_j f \ \mathrm{d}^3\boldsymbol{v} = m\delta_{ij} \int c^2 f \ \mathrm{d}^3\boldsymbol{v} = \delta_{ij} p \tag{28}$$

so that the divergence of the pressure tensor reduces to

$$\boldsymbol{\nabla} \cdot \mathbb{P} = \boldsymbol{\nabla} \cdot (\delta_{ij} p) = \boldsymbol{\nabla} p. \tag{29}$$

This assumption fails in weakly collisional or collisionless plasmas, where $f \neq f_M$, and pressure anisotropy and heat flux arise in the fluid model. At this stage the p itself has not yet been specified. One could derive an evolution equation for p by taking higher moments, but in ideal MHD the system is instead closed by prescribing an equation of state relating the p to the ρ. In what follows, and especially in our linear theory, we adopt a barotropic closure

$$p = p(\rho), \tag{30}$$

so that p fluctuations are directly related to ρ fluctuations.

3 Boltzmann equation for a non-relativistic, magnetized plasma

3.1 From Boltzmann equation to an ideal MHD fluid

A convenient starting point for magnetized fluids is kinetic theory. For a plasma (e.g. ions i and electrons e), each species has a distribution function $f_s(\boldsymbol{x}, \boldsymbol{v}, t)$ (mass m_s, charge q_s) that obeys the Boltzmann equation

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f_s + \frac{q_s}{m_s} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} f_s = C_s[f_{s'}] \,, \tag{31}$$

where $\boldsymbol{\nabla}_{\boldsymbol{v}}$ denotes the gradient in velocity space and C_s is the collision operator (which may couple species). The electromagnetic fields satisfy Maxwell's equations; in the non-relativistic, low-frequency MHD limit we will mainly use Faraday's law

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{32}$$

and (later) Ampère's law without displacement current. In *ideal* MHD we will eliminate $\boldsymbol{E}$ using ideal Ohm's law, so the evolution can be written entirely in terms of $\boldsymbol{B}$ and the bulk flow $\boldsymbol{u}$.

3.1.1 Moments of the multi-species Boltzmann equation

We have the usual velocity moments for each species, with peculiar velocity $\boldsymbol{c}_s \equiv \boldsymbol{v} - \boldsymbol{u}_s$, e.g.,

$$n_s \equiv \int f_s \ \mathrm{d}^3\boldsymbol{v}, \qquad n_s \boldsymbol{u}_s \equiv \int \boldsymbol{v} f_s \ \mathrm{d}^3\boldsymbol{v}, \tag{33}$$

$$\mathbb{P}_s \equiv m_s \int \boldsymbol{c}_s \otimes \boldsymbol{c}_s \, f_s \ \mathrm{d}^3\boldsymbol{v}, \qquad \mathbb{Q}_s \equiv m_s \int \boldsymbol{c}_s \otimes \boldsymbol{c}_s \otimes \boldsymbol{c}_s \, f_s \ \mathrm{d}^3\boldsymbol{v}. \tag{34}$$

$$\dots \tag{35}$$

as previously stated for the hydrodynamic case.

Continuity equation (zeroth moment). Integrating (31) over $\boldsymbol{v}$ gives number conservation for each species,

$$\frac{\partial n_s}{\partial t} + \boldsymbol{\nabla} \cdot (n_s \boldsymbol{u}_s) = 0. \tag{36}$$

Momentum equation (first moment). Multiplying (31) by $m_s \boldsymbol{v}$ and integrating over $\boldsymbol{v}$ yields

$$m_s n_s \left(\frac{\partial \boldsymbol{u}_s}{\partial t} + \boldsymbol{u}_s \cdot \boldsymbol{\nabla} \boldsymbol{u}_s \right) = q_s n_s \left(\boldsymbol{E} + \boldsymbol{u}_s \times \boldsymbol{B} \right) - \boldsymbol{\nabla} \cdot \mathbb{P}_s + \mathbf{R}_s, \tag{37}$$

where $\mathbf{R}_s \equiv \int m_s \boldsymbol{v}\, C_s \, \mathrm{d}^3\boldsymbol{v}$ is the collisional momentum exchange. For a number of processes, e.g., ion-neutral damping,

$$\mathbf{R}_1 \propto n_1 n_2 (\boldsymbol{u}_1 - \boldsymbol{u}_2), \qquad \mathbf{R}_2 \propto n_2 n_1 (\boldsymbol{u}_2 - \boldsymbol{u}_1) = -\mathbf{R}_1, \tag{38}$$

hence, $\sum_s \mathbf{R}_s = \mathbf{0}$ (Newton's third law). This is a statement that collisions exchange momentum internally but do not create or destroy total momentum (which is not always true).

Single-fluid variables. We can define the total mass density, bulk velocity, and current by summing over the total number of species in the underlying plasma,

$$\rho \equiv \sum_s m_s n_s, \qquad \boldsymbol{u} \equiv \frac{1}{\rho} \sum_s m_s n_s \boldsymbol{u}_s, \qquad \boldsymbol{J} \equiv \sum_s q_s n_s \boldsymbol{u}_s, \tag{39}$$

and the total pressure tensor $\mathbb{P} \equiv \sum_s \mathbb{P}_s$. Summing (37) over species (and using $\sum_s \mathbf{R}_s = \mathbf{0}$) gives the single-fluid momentum equation

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} - \boldsymbol{\nabla} \cdot \mathbb{P}, \tag{40}$$

where $\rho_e \equiv \sum_s q_s n_s$ is the charge density. In the MHD regime the plasma is *quasi-neutral*, $\rho_e \approx 0$, so that

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = \boldsymbol{J} \times \boldsymbol{B} - \boldsymbol{\nabla} \cdot \mathbb{P}. \tag{41}$$

One can perform the same sum over s to the continuity equation, which I state in §3.2.

Induction equation (Maxwell + Ohm). Finally we turn to the evolution of the magnetic field. Faraday's law is

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{42}$$

To close the system, we require a relation between $\boldsymbol{E}$, $\boldsymbol{B}$, and the fluid velocity. This is provided by Ohm's law, which follows from the electron momentum equation. In the simplest (non-relativistic) setting, neglecting electron inertia, electron pressure gradients, and resistivity, the electric field in the fluid frame, $\boldsymbol{E}'$ vanishes,

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \mathbf{0}. \tag{43}$$

This is the ideal Ohm's law and expresses perfect conductivity (flux freezing). Substituting (43) into Faraday's law gives the *ideal induction equation*:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E} = -\boldsymbol{\nabla} \times (-\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}). \tag{44}$$

Taking the divergence of (44) yields

$$\frac{\partial}{\partial t} (\boldsymbol{\nabla} \cdot \boldsymbol{B}) = \boldsymbol{\nabla} \cdot \left[\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \right] = 0, \tag{45}$$

so if $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ initially it remains so.

3.2 Ideal MHD equations.

Putting this altogether, the ideal MHD equations are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{46}$$

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = -\boldsymbol{\nabla} p + \boldsymbol{J} \times \boldsymbol{B}, \tag{47}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{48}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{49}$$

with $\boldsymbol{J}$ given by (50). The system is closed by an equation of state (e.g., barotropic $p = p(\rho)$, adiabatic $p\rho^{-\gamma} = \text{const}$, etc.).

3.3 The repercussions and assumptions of ideal MHD

In obtaining *ideal* MHD, our models adopts the following:

1. **Fast (normally collisional) relaxation to LTE:** collisions rapidly drive each species toward a local thermal equilibrium (LTE) that relaxes f towards the Maxwell-Boltzmann distribution, f_M (to understand why, look into Boltzmann's H-theorem), enabling us to generate evolution equations in the way we have throughout these notes (in my Chapman-Enskog notes this is made explicit by expanding $f_s = f_{s,M} + \epsilon f_s^{(1)} + \cdots$ and deriving transport equations that account a small perturbation away from f_M, $\epsilon f_s^{(1)}$. Note that this means that the plasma we consider has to be collisional enough i.e., $\text{Kn} = \lambda_{\text{mfp}}/\text{L} \ll 1$, for a fluid theory such as ideal MHD to be justified. This is not the case in most high-energy plasmas around compact objects.

2. **Isotropic scalar pressure:** As discussed previously, for a Maxwell-Boltzmann distribution function, $\mathbb{P} = p\,\mathbb{I}$, so that $-\boldsymbol{\nabla} \cdot \mathbb{P} = -\boldsymbol{\nabla} p$. Weakly collisional plasmas, perturbed away from LTE, can have pressure anisotropy, which leads to additional fluid operators (Braginskii) and evolution equations (CGL). For collisionless plasmas, all bets are off.

3. **No heat flux:** One can show explicitly that for an Maxwell-Boltzmann distribution, the heat-flux tensor, $\mathbb{Q}_s$ is exactly zero (there exists no third-order moments because f is perfectly symmetric). Hence there is no flux of heat between fluid elements in the ideal approximation (of course... everything is LTE).

4. **Quasi-neutrality:** We assumed that $\rho_e = 0$, which allowed us to eliminate E from the momentum equation. This is not Lorentz invariant, and so for relativistic plasmas the nature of $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho_e$ is frame-dependent.

5. **No displacement current:** We neglect displacement current so that Ampère's law reduces to

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \qquad \left(\text{instead of } \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} \right). \tag{50}$$

   Why can we do this? For characteristic length scales L, $E \sim UB$ (ideal Ohm's law), timescales $T \sim L/U$, $J \sim B/(\mu_0 L)$, $1/c^2 \sim (\mu_0 \varepsilon_0)$ then

$$\frac{|\varepsilon_0 \mu_0\, \partial_t E|}{|\mu_0 J|} \sim \frac{(1/c^2) U^2 B/L}{B/L} \sim \frac{U^2}{c^2} \tag{51}$$

   where U is the fastest signal speed in the plasma (the fast magnetosonic speed, which we derive later). This speed is $U \ll c$ in the non-relativistic limit, making $\varepsilon_0 \mu_0\, \partial_t E$ negligible in Ampère's law, and simplifying our Lorentz force. This approximation removes vacuum electromagnetic waves ($\omega = ck$) from the model, leaving only plasma-supported modes.

6. **Infinite conductivity (ideal Ohm):** we have neglected so many terms in the electron momentum equation that the electric field in the fluid frame must vanish,

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = 0, \tag{52}$$

i.e., ideal Ohm's law. This corresponds to perfect "flux freezing" (the mass density and magnetic fields are frozen together in the plasma – fields lines do not slip through the plasma).

7. **Single-fluid closure:** We assume that species in the underlying plasma are sufficiently coupled to be described by a single bulk velocity.

4 Linearization

4.1 Small perturbations about a homogeneous equilibrium

We linearize ideal MHD about a homogeneous, stationary background with constant density, pressure, and magnetic field,

$$\rho(\boldsymbol{x}, t) = \rho_0 + \epsilon \rho_1(\boldsymbol{x}, t), \tag{53}$$
$$p(\boldsymbol{x}, t) = p_0 + \epsilon p_1(\boldsymbol{x}, t), \tag{54}$$
$$\boldsymbol{u}(\boldsymbol{x}, t) = \epsilon \boldsymbol{u}_1(\boldsymbol{x}, t), \tag{55}$$
$$\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{B}_0 + \epsilon \boldsymbol{B}_1(\boldsymbol{x}, t), \tag{56}$$

where $\epsilon \ll 1$ is a small parameter, meaning, e.g., $\epsilon \rho_1 \ll \rho_0$, etc. The background state satisfies

$$\rho_0,\ p_0,\ \boldsymbol{B}_0 = \text{const.}, \qquad \boldsymbol{u}_0 = \boldsymbol{0}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B}_0 = 0. \tag{57}$$

We assume a barotropic/adiabatic equation of state closure so that to linear order in ϵ

$$p_1 = c_s^2 \, \rho_1, \qquad c_s^2 \equiv \left. \frac{\partial p}{\partial \rho} \right|_0 . \tag{58}$$

i.e., the linear order closure for the equation of state is an isothermal one, since $T \propto \sqrt{c_s}$.

4.2 Linearization of the ideal MHD equations

We substitute the perturbation expansions into (46)–(49) and retain only terms up to $\mathcal{O}(\epsilon)$. Throughout, we use that the background is uniform and stationary,

$$\frac{\partial}{\partial t}\rho_0 = \frac{\partial}{\partial t}p_0 = \frac{\partial}{\partial t}\boldsymbol{B}_0 = \boldsymbol{0}, \tag{59}$$

and

$$\boldsymbol{\nabla}\rho_0 = \boldsymbol{\nabla}p_0 = \boldsymbol{\nabla} \otimes \boldsymbol{B}_0 = \boldsymbol{0}. \tag{60}$$

Let us now linearize the MHD equations, one-by-one.

4.2.1 Linearized Continuity Equation

$$0 = \frac{\partial}{\partial t}\left(\rho_0 + \epsilon\rho_1\right) + \boldsymbol{\nabla} \cdot \left[\left(\rho_0 + \epsilon\rho_1\right)\left(\epsilon \boldsymbol{u}_1\right)\right] \tag{61}$$
$$= \epsilon \frac{\partial \rho_1}{\partial t} + \epsilon\, \boldsymbol{\nabla} \cdot \left(\rho_0 \boldsymbol{u}_1\right) + \mathcal{O}(\epsilon^2). \tag{62}$$

Dividing by ϵ and discarding higher-order terms gives

$$\frac{\partial \rho_1}{\partial t} + \rho_0\, \boldsymbol{\nabla} \cdot \boldsymbol{u}_1 = 0. \tag{63}$$

4.2.2 Linearized Induction Equation

$$\frac{\partial}{\partial t}(\boldsymbol{B}_0 + \epsilon \boldsymbol{B}_1) = \boldsymbol{\nabla} \times [(\epsilon \boldsymbol{u}_1) \times (\boldsymbol{B}_0 + \epsilon \boldsymbol{B}_1)] \tag{64}$$

$$\epsilon \frac{\partial \boldsymbol{B}_1}{\partial t} = \epsilon\, \boldsymbol{\nabla} \times (\boldsymbol{u}_1 \times \boldsymbol{B}_0) + \mathcal{O}(\epsilon^2), \tag{65}$$

so

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u}_1 \times \boldsymbol{B}_0). \tag{66}$$

Taking the divergence gives

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{B}_1) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times (\boldsymbol{u}_1 \times \boldsymbol{B}_0)) = 0, \tag{67}$$

so if $\boldsymbol{\nabla} \cdot \boldsymbol{B}_1 = 0$ initially, it remains so,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}_1 = 0, \tag{68}$$

which is very sensible.

4.2.3 Linearized Momentum Equation

The inertial term becomes

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = (\rho_0 + \epsilon \rho_1) \left(\epsilon \frac{\partial \boldsymbol{u}_1}{\partial t} + \epsilon^2 \boldsymbol{u}_1 \cdot \boldsymbol{\nabla} \boldsymbol{u}_1 \right) \tag{69}$$

$$= \epsilon \rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} + \mathcal{O}(\epsilon^2), \tag{70}$$

so nonlinear advection is higher order. The pressure gradient is $-\boldsymbol{\nabla} p = -\epsilon \boldsymbol{\nabla} p_1$. For the Lorentz force,

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = (\boldsymbol{\nabla} \times (\boldsymbol{B}_0 + \epsilon \boldsymbol{B}_1)) \times (\boldsymbol{B}_0 + \epsilon \boldsymbol{B}_1) \tag{71}$$

$$= \epsilon (\boldsymbol{\nabla} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 + \mathcal{O}(\epsilon^2), \tag{72}$$

since $\boldsymbol{\nabla} \times \boldsymbol{B}_0 = \boldsymbol{0}$. Collecting $\mathcal{O}(\epsilon)$ terms gives

$$\rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} = -\boldsymbol{\nabla} p_1 + \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0. \tag{73}$$

It is often useful to rewrite the Lorentz force using vector identities. For constant $\boldsymbol{B}_0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{B}_1 = 0$,

$$(\boldsymbol{\nabla} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 = (\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}) \boldsymbol{B}_1 - \boldsymbol{\nabla} (\boldsymbol{B}_0 \cdot \boldsymbol{B}_1), \tag{74}$$

so that

$$\rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} = -\boldsymbol{\nabla} \left(p_1 + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{B}_1}{\mu_0} \right) + \frac{1}{\mu_0} (\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}) \boldsymbol{B}_1. \tag{75}$$

The first term is the gradient of the total (thermal + magnetic) pressure perturbation, and the second term is magnetic tension along the background field.

5 Plane-wave solutions

We now seek normal-mode solutions of the linearized MHD equations. We therefore assume solutions of the form

$$\rho_1(\boldsymbol{x}, t) = \hat{\rho}\, e^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)}, \tag{76}$$

$$p_1(\boldsymbol{x}, t) = \hat{p}\, e^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)}, \tag{77}$$

$$\boldsymbol{u}_1(\boldsymbol{x}, t) = \hat{\boldsymbol{u}}\, e^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)}, \tag{78}$$

$$\boldsymbol{B}_1(\boldsymbol{x}, t) = \hat{\boldsymbol{B}}\, e^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)}, \tag{79}$$

where $\boldsymbol{k}$ is the wavevector, ω is the (complex) frequency, and hatted quantities denote constant complex amplitudes. Under this ansatz, temporal and spatial derivatives become

$$\frac{\partial}{\partial t} \to -i\omega, \qquad \boldsymbol{\nabla} \to i\boldsymbol{k}, \tag{80}$$

so the linearized system of partial differential equations reduces to a system of algebraic equations.

5.1 Constructing the linear system.

The goal is now to construct a linear system for the wave momentum ampltidues. Hence we are going to focus on building each term from the linear momentum equation. The linearized continuity equation, Eq. (63), becomes

$$-\omega\hat{\rho} + \rho_0\, \boldsymbol{k} \cdot \hat{\boldsymbol{u}} = 0. \tag{81}$$

The induction equation, Eq. (66), becomes

$$-\omega\hat{\boldsymbol{B}} = \boldsymbol{k} \times (\hat{\boldsymbol{u}} \times \boldsymbol{B}_0). \tag{82}$$

The momentum equation, Eq. (73), becomes

$$-\omega\rho_0\hat{\boldsymbol{u}} = -\boldsymbol{k}\hat{p} + \frac{1}{\mu_0}(\boldsymbol{k} \times \hat{\boldsymbol{B}}) \times \boldsymbol{B}_0. \tag{83}$$

Finally, the solenoidal constraint and the equation of state give

$$\boldsymbol{k} \cdot \hat{\boldsymbol{B}} = 0, \qquad \hat{p} = c_s^2\hat{\rho}. \tag{84}$$

Equations (81)–(84) form a closed algebraic system for the unknown amplitudes $(\hat{\rho}, \hat{p}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{B}})$, some of which we can derive immediately, which I shall do below.

5.1.1 Constructing the eigenvalue problem for $\hat{\boldsymbol{u}}$

After eliminating $\hat{p}$ using (88) and $\hat{\boldsymbol{B}}$ using (90), the momentum equation can be written as the eigenvalue problem

$$\mathbb{M}(\boldsymbol{k}) \cdot \hat{\boldsymbol{u}} = \omega^2\hat{\boldsymbol{u}}, \tag{85}$$

where $\mathbb{M}$ is the 3×3 matrix of amplitudes for $\hat{\boldsymbol{u}}$.

5.1.2 Rearrange $\hat{\boldsymbol{u}}$.

To construct this, we start with the Fourier–space momentum equation,

$$-\omega\rho_0\hat{\boldsymbol{u}} = -\boldsymbol{k}\hat{p} + \frac{1}{\mu_0}(\boldsymbol{k} \times \hat{\boldsymbol{B}}) \times \boldsymbol{B}_0. \tag{86}$$

and multiply by $-\omega$ to give,

$$\boxed{\omega^2\rho_0\hat{\boldsymbol{u}} = \omega\boldsymbol{k}\hat{p} - \frac{\omega}{\mu_0}(\boldsymbol{k} \times \hat{\boldsymbol{B}}) \times \boldsymbol{B}_0.} \tag{87}$$

5.1.3 Rearrange $\hat{\rho}$.

From (81) and (84) we obtain

$$\boxed{\hat{p} = c_s^2\hat{\rho} = \frac{\rho_0 c_s^2}{\omega}\, \boldsymbol{k} \cdot \hat{\boldsymbol{u}}.} \tag{88}$$

5.1.4 Rearrange $\hat{\boldsymbol{B}}$.

Using the vector identity

$$\boldsymbol{k} \times (\hat{\boldsymbol{u}} \times \boldsymbol{B}_0) = (\boldsymbol{k} \cdot \boldsymbol{B}_0)\hat{\boldsymbol{u}} - (\boldsymbol{k} \cdot \hat{\boldsymbol{u}})\boldsymbol{B}_0, \tag{89}$$

the induction equation (82) yields

$$\boxed{\hat{\boldsymbol{B}} = -\frac{1}{\omega}\left[(\boldsymbol{k} \cdot \boldsymbol{B}_0)\hat{\boldsymbol{u}} - (\boldsymbol{k} \cdot \hat{\boldsymbol{u}})\boldsymbol{B}_0\right].} \tag{90}$$

5.1.5 Rebuilding $\hat{\boldsymbol{u}}$ in an appropriate form.

From our combined continuity and EOS we have Equation 88, hence the pressure term in Equation 87 becomes,

$$\omega \boldsymbol{k}\hat{p} = \rho_0 c_s^2\, \boldsymbol{k}(\boldsymbol{k} \cdot \hat{\boldsymbol{u}}) = \rho_0 c_s^2\, k_j(k_i \hat{u}_i) = \rho_0 c_s^2\, (k_j k_i)\hat{u}_i = \rho_0 c_s^2\, (\boldsymbol{k} \otimes \boldsymbol{k}) \cdot \hat{\boldsymbol{u}}, \tag{91}$$

which has the form of a tensor $k_j k_i$ contracted onto a velocity $\hat{u}_i$, as required. Now we must construct the linearised Lorentz force term, noting that

$$\frac{\omega}{\mu_0}(\boldsymbol{k} \times \hat{\boldsymbol{B}}) \times \boldsymbol{B}_0 = \frac{\omega}{\mu_0}\left[\hat{\boldsymbol{B}}(\boldsymbol{k} \cdot \boldsymbol{B}_0) - \boldsymbol{k}(\hat{\boldsymbol{B}} \cdot \boldsymbol{B}_0)\right], \tag{92}$$

which means we need to construct both $\hat{\boldsymbol{B}}(\boldsymbol{k} \cdot \boldsymbol{B}_0)$ and $\boldsymbol{k}(\hat{\boldsymbol{B}} \cdot \boldsymbol{B}_0)$. From the induction equation we have Equation 90, hence the first term is simply,

$$\hat{\boldsymbol{B}}(\boldsymbol{k} \cdot \boldsymbol{B}_0) = \frac{1}{\omega}\Big[(\boldsymbol{k} \cdot \boldsymbol{B}_0)^2\hat{\boldsymbol{u}} - (\boldsymbol{k} \cdot \boldsymbol{B}_0)(\boldsymbol{k} \cdot \hat{\boldsymbol{u}})\boldsymbol{B}_0\Big], \tag{93}$$

and the second term is

$$\hat{\boldsymbol{B}} \cdot \boldsymbol{B}_0 = \frac{1}{\omega}\Big[(\boldsymbol{k} \cdot \boldsymbol{B}_0)(\hat{\boldsymbol{u}} \cdot \boldsymbol{B}_0) - (\boldsymbol{k} \cdot \hat{\boldsymbol{u}})(\boldsymbol{B}_0 \cdot \boldsymbol{B}_0)\Big]. \tag{94}$$

hence,

$$\boldsymbol{k}(\hat{\boldsymbol{B}} \cdot \boldsymbol{B}_0) = \frac{1}{\omega}\Big[(\boldsymbol{k} \cdot \boldsymbol{B}_0)(\hat{\boldsymbol{u}} \cdot \boldsymbol{B}_0)\boldsymbol{k} - (\boldsymbol{k} \cdot \hat{\boldsymbol{u}})B_0^2\, \boldsymbol{k}\Big]. \tag{95}$$

Putting them together gives,

$$\frac{\omega}{\mu_0}(\boldsymbol{k} \times \hat{\boldsymbol{B}}) \times \boldsymbol{B}_0 = \frac{1}{\mu_0}\Big[(\boldsymbol{k} \cdot \boldsymbol{B}_0)^2\hat{\boldsymbol{u}} - (\boldsymbol{k} \cdot \boldsymbol{B}_0)(\boldsymbol{k} \cdot \hat{\boldsymbol{u}})\boldsymbol{B}_0 - (\boldsymbol{k} \cdot \boldsymbol{B}_0)(\hat{\boldsymbol{u}} \cdot \boldsymbol{B}_0)\boldsymbol{k} + (\boldsymbol{k} \cdot \hat{\boldsymbol{u}})B_0^2\, \boldsymbol{k}\Big]. \tag{96}$$

Now, substituting this back into the momentum equation, and with some small rearrangements, we get,

$$\boxed{\omega^2\rho_0\hat{\boldsymbol{u}} = \rho_0 c_s^2\,(\boldsymbol{k} \otimes \boldsymbol{k}) \cdot \hat{\boldsymbol{u}} - \frac{1}{\mu_0}\Big[(\boldsymbol{k} \cdot \boldsymbol{B}_0)^2\hat{\boldsymbol{u}} - (\boldsymbol{k} \cdot \hat{\boldsymbol{u}})(\boldsymbol{k} \cdot \boldsymbol{B}_0)\boldsymbol{B}_0 - (\boldsymbol{k} \cdot \boldsymbol{B}_0)(\hat{\boldsymbol{u}} \cdot \boldsymbol{B}_0)\boldsymbol{k} + B_0^2(\boldsymbol{k} \cdot \hat{\boldsymbol{u}})\,\boldsymbol{k}\Big].} \tag{97}$$

Next, we define $v_A^2 \equiv B_0^2/(\mu_0\rho_0)$, $\hat{\boldsymbol{b}}_0 \equiv \boldsymbol{B}_0/B_0$, and $k_\parallel \equiv \boldsymbol{k} \cdot \hat{\boldsymbol{b}}_0$, which gives

$$\frac{(\boldsymbol{k} \cdot \boldsymbol{B}_0)^2}{\mu_0\rho_0} = v_A^2(\boldsymbol{k} \cdot \hat{\boldsymbol{b}}_0)^2 = v_A^2 k_\parallel^2, \tag{98}$$

$$\frac{(\boldsymbol{k} \cdot \boldsymbol{B}_0)\boldsymbol{B}_0}{\mu_0\rho_0} = v_A^2 k_\parallel \hat{\boldsymbol{b}}_0, \tag{99}$$

$$\frac{(\boldsymbol{k} \cdot \boldsymbol{B}_0)(\hat{\boldsymbol{u}} \cdot \boldsymbol{B}_0)}{\rho_0\mu_0} = v_A^2 k_\parallel(\hat{\boldsymbol{b}}_0 \cdot \hat{\boldsymbol{u}}). \tag{100}$$

Hence, by dividing the momentum equation by ρ_0, we may write

$$\omega^2 \hat{\boldsymbol{u}} = (c_s^2 + v_A^2)\,(\boldsymbol{k}\otimes\boldsymbol{k})\cdot\hat{\boldsymbol{u}} - v_A^2 k_\parallel^2 \hat{\boldsymbol{u}} - v_A^2 k_\parallel \left(\hat{\boldsymbol{b}}_0(\boldsymbol{k}\cdot\hat{\boldsymbol{u}}) + \boldsymbol{k}(\hat{\boldsymbol{b}}_0\cdot\hat{\boldsymbol{u}})\right). \tag{101}$$

and by undotting $\hat{\boldsymbol{u}}$,

$$\omega^2 \hat{\boldsymbol{u}} = \overbrace{\left[(c_s^2 + v_A^2)\,\boldsymbol{k}\otimes\boldsymbol{k} - v_A^2 k_\parallel^2 \mathbb{I} - v_A^2 k_\parallel \left(\hat{\boldsymbol{b}}_0\otimes\boldsymbol{k} + \boldsymbol{k}\otimes\hat{\boldsymbol{b}}_0\right)\right]}^{\mathbb{M}(\boldsymbol{k})} \cdot\hat{\boldsymbol{u}}. \tag{102}$$

or in index notation,

$$M_{ij}(k_l) = v_A^2 k_\parallel^2\,\delta_{ij} + (c_s^2 + v_A^2)\,k_i k_j - v_A^2 k_\parallel \left(k_i \hat{b}_{0j} + \hat{b}_{0i} k_j\right), \tag{103}$$

which is real and symmetric.

6 Ideal MHD Eigenmodes

We now solve the eigenvalue problem

$$\mathbb{M}(\boldsymbol{k})\cdot\hat{\boldsymbol{u}} = \omega^2\hat{\boldsymbol{u}}, \tag{104}$$

derived in the previous section, and interpret the resulting eigenvalues and eigenvectors as physical wave modes of the magnetized fluid. We choose coordinates such that the background magnetic field defines the z-axis,

$$\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}, \tag{105}$$

and take the wavevector to lie in the x-z plane,

$$\boldsymbol{k} = (k_\perp,\, 0,\, k_\parallel), \tag{106}$$

so that $k_\parallel = \boldsymbol{k}\cdot\hat{\boldsymbol{b}}_0$ and $k^2 = k_\perp^2 + k_\parallel^2$. In this basis, the eigenvalue problem becomes

$$\mathbb{M}(\boldsymbol{k}) = \begin{pmatrix} (c_s^2 + v_A^2)k_\perp^2 + v_A^2 k_\parallel^2 & 0 & c_s^2 k_\perp k_\parallel \\ 0 & v_A^2 k_\parallel^2 & 0 \\ c_s^2 k_\perp k_\parallel & 0 & c_s^2 k_\parallel^2 \end{pmatrix}. \tag{107}$$

This system immediately decouples into a transverse mode (y-direction) and two coupled compressive modes in the (x, z) plane.

6.1 Alfvén mode

The y-component of the eigenvalue problem satisfies

$$\omega^2 \hat{u}_y = v_A^2 k_\parallel^2 \hat{u}_y, \tag{108}$$

giving the dispersion relation

$$\boxed{\omega = \pm v_A k_\parallel,} \tag{109}$$

i.e. the waves propagate only along the direction of the background magnetic field, with phase speed equal to the Alfvén speed v_A. The corresponding eigenvector is

$$\hat{\boldsymbol{u}} = \hat{u}_y\hat{\boldsymbol{y}}. \tag{110}$$

Because $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$ and $\hat{\boldsymbol{u}} = (0, \hat{u}_y, 0)$,

$$\boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0, \qquad \hat{\boldsymbol{u}}\cdot\boldsymbol{B}_0 = 0. \tag{111}$$

Compressibility. From the linearized continuity equation,

$$\hat{\rho} = \frac{\rho_0}{\omega}\, \boldsymbol{k} \cdot \hat{\boldsymbol{u}}, \tag{112}$$

we immediately see that $\hat{\rho} = 0$ for the Alfvén mode, and hence $\hat{p} = 0$ via the EOS. The Alfvén wave is therefore incompressible and purely transverse.

Magnetic field fluctuations The associated magnetic-field fluctuations follow directly from Eq. (90),

$$\hat{\boldsymbol{B}} = -\frac{1}{\omega}\Big[(\boldsymbol{k} \cdot \boldsymbol{B}_0)\hat{\boldsymbol{u}} - (\boldsymbol{k} \cdot \hat{\boldsymbol{u}})\boldsymbol{B}_0\Big] = -\frac{\boldsymbol{k} \cdot \boldsymbol{B}_0}{\omega}\hat{\boldsymbol{u}}. \tag{113}$$

Using $\boldsymbol{k} \cdot \boldsymbol{B}_0 = k_\parallel B_0$ and $\omega = \pm v_A k_\parallel$, this becomes

$$\hat{\boldsymbol{B}} = \mp\frac{B_0}{v_A}\hat{\boldsymbol{u}} = \mp\sqrt{\mu_0\rho_0}\,\hat{\boldsymbol{u}}. \tag{114}$$

Thus the velocity and magnetic perturbations are exactly parallel (or anti-parallel), and both are perpendicular to $\boldsymbol{k}$ and $\boldsymbol{B}_0$. In velocity units,

$$\frac{\hat{\boldsymbol{B}}}{\sqrt{\mu_0\rho_0}} = \mp\hat{\boldsymbol{u}}, \tag{115}$$

which is the standard Elsässer relation for Alfvén waves (not covered in this lecture, but this form is a building block for Alfvénic turbulence theory, where wavepackets, $\boldsymbol{z}^\pm = \boldsymbol{u} \pm \boldsymbol{B}/\sqrt{\mu_0\rho_0}$, are shown to be exact nonlinear solutions to the incompressible MHD equations).

Restoring force. Finally, writing the linearized momentum equation in pressure-tension form,

$$\rho_0\partial_t\boldsymbol{u}_1 = -\boldsymbol{\nabla}\left(p_1 + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{B}_1}{\mu_0}\right) + \frac{1}{\mu_0}(\boldsymbol{B}_0 \cdot \boldsymbol{\nabla})\boldsymbol{B}_1, \tag{116}$$

we note that both $p_1 = 0$ and $\boldsymbol{B}_0 \cdot \boldsymbol{B}_1 = 0$ for the Alfvén mode, so the total pressure gradient vanishes. The only remaining restoring force is magnetic tension.

6.2 Fast and slow magnetosonic modes.

The remaining two eigenmodes correspond to velocity perturbations lying in the xz plane spanned by $\boldsymbol{k}$ and $\boldsymbol{B}_0$. These are obtained by solving the reduced 2×2 eigenvalue problem

$$\begin{pmatrix} (c_s^2 + v_A^2)k_\perp^2 + v_A^2 k_\parallel^2 & c_s^2 k_\perp k_\parallel \\ c_s^2 k_\perp k_\parallel & c_s^2 k_\parallel^2 \end{pmatrix} \begin{pmatrix} \hat{u}_x \\ \hat{u}_z \end{pmatrix} = \omega^2 \begin{pmatrix} \hat{u}_x \\ \hat{u}_z \end{pmatrix}. \tag{117}$$

The characteristic equation is

$$\omega^4 - \omega^2 k^2 (c_s^2 + v_A^2) + c_s^2 v_A^2 k^2 k_\parallel^2 = 0, \tag{118}$$

with solutions

$$\boxed{\omega^2 = \frac{k^2}{2}\Big[(c_s^2 + v_A^2) \pm \sqrt{(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2\theta}\Big],} \tag{119}$$

where

$$\cos\theta = \frac{k_\parallel}{k}, \qquad \text{since} \qquad k_\parallel = \frac{kB_0\cos\theta}{B_0}. \tag{120}$$

The upper (+) branch corresponds to the fast magnetosonic mode, while the lower (−) branch corresponds to the slow magnetosonic mode.

Compressibility. For both magnetosonic modes, the velocity eigenvector satisfies

$$\boldsymbol{k}\cdot\hat{\boldsymbol{u}} \neq 0. \tag{121}$$

From the linearized continuity equation,

$$\hat{\rho} = \frac{\rho_0}{\omega}\,\boldsymbol{k}\cdot\hat{\boldsymbol{u}}, \tag{122}$$

it follows immediately that

$$\hat{\rho} \neq 0, \qquad \hat{p} = c_s^2\hat{\rho} \neq 0. \tag{123}$$

Thus both magnetosonic modes are compressible and involve coupled fluctuations of velocity, density, pressure, and magnetic field.

Magnetic-field fluctuations. The magnetic perturbations follow from the induction equation,

$$\hat{\boldsymbol{B}} = -\frac{1}{\omega}\Big[(\boldsymbol{k}\cdot\boldsymbol{B}_0)\hat{\boldsymbol{u}} - (\boldsymbol{k}\cdot\hat{\boldsymbol{u}})\boldsymbol{B}_0\Big]. \tag{124}$$

Unlike the Alfvén mode, both terms contribute for magnetosonic waves. The term proportional to $(\boldsymbol{k}\cdot\boldsymbol{B}_0)\hat{\boldsymbol{u}}$ represents magnetic tension due to bending of field lines, the term proportional to $(\boldsymbol{k}\cdot\hat{\boldsymbol{u}})\boldsymbol{B}_0$ represents magnetic-pressure fluctuations associated with compression of the field. As a result, $\hat{\boldsymbol{B}}$ lies in the $(\boldsymbol{k},\boldsymbol{B}_0)$ plane (xz-plane in our lecture) and is neither parallel nor perpendicular to $\hat{\boldsymbol{u}}$, in general.

Restoring forces. Writing the linearized momentum equation in pressure–tension form,

$$\rho_0\partial_t\boldsymbol{u}_1 = -\boldsymbol{\nabla}\left(p_1 + \frac{\boldsymbol{B}_0\cdot\boldsymbol{B}_1}{\mu_0}\right) + \frac{1}{\mu_0}(\boldsymbol{B}_0\cdot\boldsymbol{\nabla})\boldsymbol{B}_1, \tag{125}$$

we see that magnetosonic waves are restored by a combination of thermal pressure gradients ($\boldsymbol{\nabla}p_1$), magnetic pressure gradients ($\boldsymbol{\nabla}(\boldsymbol{B}_0\cdot\boldsymbol{B}_1)$), magnetic tension along the background field. The relative importance of these restoring forces depends on the propagation angle θ and on the ratio c_s/v_A.

6.2.1 Limiting cases for propagation angle.

Let us finally consider a few different limiting cases for the magnetosonic modes to gather some intuition about their propagation behavior.

Parallel propagation ($\theta = 0$):

$$\omega_{\text{fast}}^2 = k^2\max(c_s^2, v_A^2), \tag{126}$$

$$\omega_{\text{slow}}^2 = k^2\min(c_s^2, v_A^2). \tag{127}$$

The fast mode reduces to the faster of the acoustic or Alfvénic response, while the slow mode propagates at the slower characteristic speed.

Perpendicular propagation ($\theta = \pi/2$):

$$\omega_{\text{fast}}^2 = k^2(c_s^2 + v_A^2), \tag{128}$$

$$\omega_{\text{slow}}^2 = 0. \tag{129}$$

In this case the slow mode becomes non-propagating, while the fast mode propagates isotropically in the plane perpendicular to $\boldsymbol{B}_0$, with effective sound speed $\sqrt{c_s^2+v_A^2}$. Thus the fast magnetosonic mode is a compressive modes that propagates in all angles with respect to the background field $\omega^2 \propto k^2$, whereas the slow mode is field-guided $\theta = \pi/2 \implies \omega = 0 = k$, i.e., the slow mode no longer defines a planar wave at $\theta = \pi/2$. We can see what this mode does by substituting its properties back in our linear momentum,

$$\rho_0\partial_t\boldsymbol{u}_1 = -\boldsymbol{\nabla}\left(p_1 + \frac{\boldsymbol{B}_0\cdot\boldsymbol{B}_1}{\mu_0}\right) + \frac{1}{\mu_0}(\boldsymbol{B}_0\cdot\boldsymbol{\nabla})\boldsymbol{B}_1, \tag{130}$$

where $\omega = 0 \Rightarrow \partial_t \boldsymbol{u}_1 = 0$ and $k_\parallel = 0 \Rightarrow (\boldsymbol{B}_0 \cdot \nabla)\boldsymbol{B}_1 = 0$, hence

$$\boldsymbol{\nabla} p_1 = -\boldsymbol{\nabla} \frac{\boldsymbol{B}_0 \cdot \boldsymbol{B}_1}{\mu_0} \tag{131}$$

is a structure in pressure-balance. Hence the $\theta = \pi/2$ slow mode describes a non-propagating structure in pressure equilibrium (see, e.g., pressure-balanced structures (PBSs) observed in the solar wind).

6.2.2 High- and low-β limits.

Additional insight is obtained by considering the relative importance of thermal and magnetic pressure, characterized by the plasma beta $\beta \sim c_s^2/v_A^2$.

Low-β plasma ($v_A \gg c_s$):

$$\omega_{\text{fast}}^2 \simeq v_A^2 k^2, \tag{132}$$

$$\omega_{\text{slow}}^2 \simeq c_s^2 k_\parallel^2. \tag{133}$$

In this regime the fast magnetosonic mode is predominantly magnetic, propagating at nearly the Alfvén speed in all angles, while the slow mode behaves as a field-guided acoustic wave and remains non-propagating for perpendicular propagation.

High-β plasma ($c_s \gg v_A$):

$$\omega_{\text{fast}}^2 \simeq c_s^2 k^2, \tag{134}$$

$$\omega_{\text{slow}}^2 \simeq v_A^2 k_\parallel^2. \tag{135}$$

In this limit the fast mode reduces to an ordinary sound wave that is only weakly influenced by the magnetic field, while the slow mode becomes a weak, magnetically dominated, field-guided compressive wave. Thus the fast mode corresponds to the dominant restoring force in the plasma (thermal or magnetic), while the slow mode represents the subdominant compressive response constrained to propagate along the magnetic field.